Criterion for deciding zeta-like multizeta values in positive characteristic

Yen-Liang Kuan

National Taiwan University and TIMS

Taipei-Xian Number Theory Workshop
October 15, 2015

- Joint work with Huei-Jeng Chen and Yi-Hsuan Lin.

Multiple zeta values

The multiple zeta values (abbreviated as MZV's) are real numbers defined by Euler: for any r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $s_1 \geq 2$,

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Here r is the depth, $w = s_1 + \cdots + s_r$ is the weight of $\zeta(s_1, \ldots, s_r)$.

These values are generalizations of the Riemann zeta function at positive integers. Euler proved some relations between zeta and multiple zeta values, such as $\zeta(2,1) = \zeta(3)$. For the special values of Riemann zeta function at even integers, Euler's celebrated formula asserts that for $m \in \mathbb{N}$,

$$\zeta(2m) = \frac{-B_{2m}(2\pi\sqrt{-1})^{2m}}{2(2m)!},$$

where $B_{2m} \in \mathbb{Q}$ are Bernoulli numbers.

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Multiple zeta values

Call weight w MZV **Eulerian** if it is a rational multiple of $(2\pi\sqrt{-1})^w$, e.g. $\zeta(2m, 2m)$ in depth 2 and $\zeta(2, \ldots, 2)$ of arbitrary depth. In the depth 1 case, Euler's formula implies that the Riemann zeta value $\zeta(w)$ is Eulerian if and only if w is even.

More generally, call given weight w MZV **zeta-like** if it is a rational multiple of $\zeta(w)$.

Question : Find rules describing all Eulerian and zeta-like MZV's ?

It seems difficult to determine whether any given MZV is zeta-like or non zeta-like. In particular, Francis Brown (2012) gave a sufficient condition for zeta-like MZV's using the theory of motivic multiple zeta values. Brown's theorem implies that $\zeta(3, 1, \ldots, 3, 1)$ is a rational multiple of a power of $2\pi\sqrt{-1}$, that was conjectured by Zagier.

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Multiple zeta values in positive characteristic

Notations :

A : the polynomial ring $\mathbb{F}_q[\theta]$.

A_+ : the set of monic polynomials in A.

K : the fraction field $\mathbb{F}_q(\theta)$ of A.

∞ : the infinite place corresponding to $1/\theta$.

$K_\infty := \mathbb{F}_q((\frac{1}{\theta}))$, that is the completion of K at ∞.

$\overline{K_\infty}$: a fixed algebraic closure of K_∞.

$\mathbb{C}_\infty$: the completion of $\overline{K_\infty}$ with respect to ∞.

Carlitz zeta values, for $n \geq 1$,

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty.$$

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Multiple zeta values in positive characteristic

Let τ be the Frobenius element defined by $\tau(x) = x^q$ and we denote by $K\{\tau\}$ the twisted polynomial ring with the relation $\tau a = a^q \tau$ for all $a \in K$.

A *Carlitz module* $\mathbf{C}$ is the $\mathbb{F}_q$-algebra homomorphism

$$\mathbf{C} : \mathbb{F}_q[t] \to K\{\tau\}, \quad a \mapsto C_a,$$

characterized by $\mathbf{C}_t = \theta + \tau$.

Define $D_0 = 1$ and $D_i = \prod_{j=0}^{i-1} \left(\theta^{q^i} - \theta^{q^j} \right)$ for $i \in \mathbb{N}$. For an integer $n \geq 0$, we express n as

$$n = \sum_{i=0}^{\infty} n_i q^i \quad (0 \leq n_i \leq q-1, \; n_i = 0 \text{ for } i \gg 0),$$

and the arithmetic Γ-function is defined by

$$\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i}.$$

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The *Carlitz exponential function* is defined by

$$\exp_{\mathbf{C}}(z) = \sum_{i \geq 0} \frac{z^{q^i}}{D_i}.$$

One has the following exact sequence

$$0 \longrightarrow \tilde{\pi} A \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_{\mathbf{C}}} \mathbb{C}_\infty \longrightarrow 0.$$

Here $\tilde{\pi}$ is a *fundamental period of* $\mathbf{C}$ given by

$$\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1},$$

where $(-\theta)^{\frac{1}{q-1}}$ is a fixed choice of $(q-1)$-th root of $-\theta$.

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We write

$$\frac{z}{\exp_{\mathbf{C}}(z)} = \sum_{n \geq 0} \frac{\mathrm{BC}(n)}{\Gamma_{n+1}} z^n,$$

where $\mathrm{BC}(n) \in K$ are called Bernoulli-Carlitz numbers.

Carlitz derived an analogue of Euler's formula for $\zeta_A(n)$. He showed that if $(q-1) \mid n$, then one has

$$\zeta_A(n) = \frac{\mathrm{BC}(n)}{\Gamma_{n+1}} \tilde{\pi}^n.$$

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Thakur called $\zeta_A(s_1, \ldots, s_r)$ *Eulerian* if $\zeta_A(s_1, \ldots, s_r)/\tilde{\pi}^w \in K$, where w is the weight of $\zeta_A(s_1, \ldots, s_r)$.

On the other hand, Thakur called $\zeta_A(s_1, \ldots, s_r)$ *zeta-like* if $\zeta_A(s_1, \ldots, s_r)/\zeta_A(w) \in K$.

It is a natural question to ask how to find all Eulerian/zeta-like MZV's.

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Lara Rodríguez and Thakur proved precise formulas for certain families of Eulerian/zeta-like MZV's, and gave conjectures on which r-tuples $(s_1, \ldots, s_r)$ may occur for zeta-like MZV's.

Chang-Papanikolas-Yu gave a criterion for Eulerian/zeta-like MZV's by constructing an abelian t-module E defined over A and integral points $\mathbf{v}_{\mathfrak{s}}$, $\mathbf{u}_{\mathfrak{s}} \in E(A)$. They proved that $\zeta_A(\mathfrak{s})$ is Eulerian (resp. zeta-like) if and only if $\mathbf{v}_{\mathfrak{s}}$ is an $\mathbb{F}_q[t]$-torsion point in $E(A)$ (resp. $\mathbf{v}_{\mathfrak{s}}, \mathbf{u}_{\mathfrak{s}}$ are $\mathbb{F}_q[t]$-linearly dependent in $E(A)$).

Today, I shall give an effective algorithm for determining whether any given MZV is zeta-like or non zeta-like, and I shall state conjectures on certain families of zeta-like MZV's.

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Frobenius modules

Given $n \in \mathbb{Z}$ and $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$, put

$$f^{(n)} = \sum_i a_i^{q^n} t^i \in \mathbb{C}_\infty((t)).$$

Let $\overline{K}[t, \sigma] = \overline{K}[t][\sigma]$ be the non-commutative $\overline{K}[t]$-algebra generated by the variable σ with the relation

$$\sigma f = f^{(-1)}\sigma, \quad f \in \overline{K}[t].$$

A *Frobenius module* is a left $\overline{K}[t, \sigma]$-module and it is free of finite rank over $\overline{K}[t]$.

For a Frobenius module M of rank r over $\overline{K}[t]$, we say that M is defined by a matrix Φ in $\mathrm{Mat}_r(\overline{K}[t])$ if the σ-action on a given $\overline{K}[t]$-basis of M is represented by the matrix Φ.

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A *Frobenius module* is a left $\overline{K}[t, \sigma]$-module and it is free of finite rank over $\overline{K}[t]$.

For a Frobenius module M of rank r over $\overline{K}[t]$, we say that M is defined by a matrix Φ in $\mathrm{Mat}_r(\overline{K}[t])$ if the σ-action on a given $\overline{K}[t]$-basis of M is represented by the matrix Φ.

Example : Let $\mathbf{1}$ be the trivial Frobenius module. The underlying space of $\mathbf{1}$ is $\overline{K}[t]$, and the action of σ on $\mathbf{1}$ is given by

$$\sigma(f) := f^{(-1)}, \quad f \in \mathbf{1}.$$

Recall that $D_0 = 1$ and $D_i = \prod_{j=0}^{i-1} \left(\theta^{q^i} - \theta^{q^j} \right)$ for $i \in \mathbb{N}$.

Put $G_0 = 1$. For $i \in \mathbb{N}$, let

$$G_i = \prod_{j=1}^{i} (t^{q^i} - \theta^{q^j}) \in \mathbb{F}_q[t, \theta].$$

For $n = 0, 1, 2, \ldots$, we define the Anderson-Thakur polynomials $H_n \in A[t]$ by the generating series identity

$$\left(1 - \sum_{i=0}^{\infty} \frac{G_i}{D_i|_{\theta=t}} x^{q^i} \right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

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Frobenius modules connected to $\zeta_A(\mathfrak{s})$

For any r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, Anderson-Thakur related $\zeta_A(\mathfrak{s})$ to a period of a Frobenius module, that is an extension of $\mathbf{1}$ by M, where M is defined by the following matrix

$$\Phi := \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & \cdots & 0 \\ H_{s_1-1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & \ddots & \cdots & 0 \\ \vdots & \ddots & (t-\theta)^{s_{r-1}+s_r} & 0 \\ 0 & \cdots & H_{s_{r-1}-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} & (t-\theta)^{s_r} \end{pmatrix}.$$

Note that Φ is a matrix in $\mathrm{Mat}_r\left(\overline{K}[t]\right)$.

Frobenius modules and t-modules

Let M be the Frobenius module defined by Φ. Anderson gave an identification between $M/(\sigma-1)M$ and a t-module (E,ρ) defined over $\overline{K}$.

We now recall the definition of t-modules.

For $d \in \mathbb{N}$, a d-dimensional *t-module* is a pair (E,ρ), where E is the d-dimensional algebraic group $\mathbb{G}_a^d$ and ρ is an $\mathbb{F}_q$-linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \to \mathrm{Mat}_d(\mathbb{C}_\infty\{\tau\})$$

such that $\rho_t = \alpha_0 + \sum_i \alpha_i \tau^i$ where $\alpha_i \in \mathrm{Mat}_d(\mathbb{C}_\infty)$, and $\alpha_0 - \theta I_d$ is a nilpotent matrix.

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We now use the Frobenius module M to construct a t-module (E, ρ). If $\{m_1, \ldots, m_r\}$ is a $\overline{K}[t]$-basis of M, then one can check that the Frobenius module M is a free left $\overline{K}[\sigma]$-module of rank

$$d := (s_1 + \cdots + s_r) + (s_2 + \cdots + s_r) + \cdots + s_r$$

with a $\overline{K}[\sigma]$-basis $\nu_1, \ldots, \nu_d$ by

$$\left\{(t-\theta)^{s_1+\cdots+s_r-1} m_1, \cdots, (t-\theta)m_1, m_1, \ldots, (t-\theta)^{s_r-1} m_r, \ldots, m_r\right\}.$$

We define a map $\delta : \overline{K}[\sigma] \to \overline{K}$ by

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Frobenius modules and t-modules

We define the map $\Delta : M \to \mathbb{G}_a^d(\overline{K})$ given by

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Then Δ is a surjective homomorphism of $\mathbb{F}_q$-vector spaces with kernel $(\sigma - 1)M$. Note that $t(\sigma - 1)M \subseteq (\sigma - 1)M$.

Let (E, ρ) be the t-module defined over $\overline{K}$ with $E(\overline{K}) = \mathbb{G}_a^d(\overline{K})$, and $\rho : \mathbb{F}_q[t] \to \mathrm{Mat}_d(\overline{K}[\tau])$ be the $\mathbb{F}_q$-linear ring homomorphism satisfying

$$\Delta(t(u_1\nu_1 + \cdots + u_d\nu_d)) = \rho_t \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} .$$

Then Anderson proved that $M/(\sigma - 1)M$ is isomorphic to $E(\overline{K})$ as $\mathbb{F}_q[t]$-modules.

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Chang-Papanikolas-Yu criterion

For any r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, Chang-Papanikolas-Yu constructed special points $\mathbf{v}_{\mathfrak{s}}, \mathbf{u}_{\mathfrak{s}} \in E(\overline{K})$ which are defined by

$$\mathbf{v}_{\mathfrak{s}} := \Delta\left(H_{s_r-1}^{(-1)}(t-\theta)^{s_r} m_r\right) \quad \text{and} \quad \mathbf{u}_{\mathfrak{s}} := \Delta\left(H_{w-1}^{(-1)}(t-\theta)^{w} m_1\right).$$

They proved that the t-module E is defined over A and the points $\mathbf{v}_{\mathfrak{s}}$ and $\mathbf{u}_{\mathfrak{s}}$ are an integral point in $E(A)$.

Chang-Papanikolas-Yu criterion

With the notations as above. Then we have

(a) If w is divisible by $q-1$, then there exists nonzero $a \in \mathbb{F}_q[t]$ so that $\zeta_A(\mathfrak{s})$ is Eulerian if and only if $\rho_a(\mathbf{v}_{\mathfrak{s}}) = 0$ in the t-module $E(A)$.

(b) If w is not divisible by $q-1$, then we have that $\zeta_A(\mathfrak{s})$ is zeta-like if and only if there exist $c, d \in \mathbb{F}_q[t]$ (not both zero) so that $\rho_c(\mathbf{v}_{\mathfrak{s}}) + \rho_d(\mathbf{u}_{\mathfrak{s}}) = 0$ in the t-module $E(A)$.

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Chang-Papanikolas-Yu criterion

Remark : The element $a \in \mathbb{F}_q[t]$ in Chang-Papanikolas-Yu criterion can be explicitly written down as follows. For any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with all s_i divisible by $q-1$, we let

$$w_i = s_{r-i} + s_{r-i+1} + \cdots + s_r$$

for $i = 1, \ldots, r-1$. We decompose

$$w_i = p^{\ell_i} n_i (q^{h_i} - 1)$$

so that $p \nmid n_i$ and h_i is the greatest integer for which $q^{h_i} - 1 \mid w_i$. Then

$$a := a_{\mathfrak{s}} = (t^{q^{h_{r-1}}} - t)^{p^{\ell_{r-1}}} \cdots (t^{q^{h_1}} - t)^{p^{\ell_1}} \frac{\Gamma_{s_r+1}}{\Gamma_{s_r}} \mathrm{den}(\mathrm{BC}(s_r))|_{\theta=t},$$

where $\mathrm{den}(\mathrm{BC}(s_r)) \in A$ denotes the monic denominator of the Bernoulli-Carlitz number $\mathrm{BC}(s_r)$ for A.

Chang-Papanikolas-Yu criterion

From the proof of Chang-Papanikolas-Yu criterion, we know that for any r-tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, $\zeta_A(\mathfrak{s})$ is zeta-like if and only if there exist $\delta_1, \cdots, \delta_r \in \overline{K}[t]$ and $c, d \in \mathbb{F}_q[t]$ such that

$$\begin{aligned} \delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1-1}^{(-1)}(t-\theta)^w + dH_{w-1}^{(-1)}(t-\theta)^w; \\ \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} + \delta_3^{(-1)} H_{s_2-1}^{(-1)}(t-\theta)^{s_2+\cdots+s_r}; \\ &\vdots \\ \delta_{r-1} &= \delta_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} + \delta_r^{(-1)} H_{s_{r-1}-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r}; \\ \delta_r &= \delta_r^{(-1)}(t-\theta)^{s_r} + cH_{s_r-1}^{(-1)}(t-\theta)^{s_r}. \end{aligned} \tag{1}$$

Furthermore, if $q-1$ does not divide w, then we have

$$\zeta_A(\mathfrak{s}) = \frac{-\Gamma_w d|_{t=\theta}}{\Gamma_{s_1} \cdots \Gamma_{s_r} c|_{t=\theta}} \zeta_A(w).$$

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Chang-Papanikolas-Yu proved that if $\zeta_A(s_1, \ldots, s_r)$ is zeta-like, then $\zeta_A(s_2, \ldots, s_r)$ is Eulerian.

Note that, let $\mathfrak{s}' = (s_2, \ldots, s_r)$, $\zeta_A(\mathfrak{s}')$ is Eulerian if and only if there exist $\delta_2, \cdots, \delta_r \in \overline{K}[t]$ and $c = a_{\mathfrak{s}'} \in \mathbb{F}_q[t]$ such that

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Note that if $(\delta_1, \ldots, \delta_r, c, d)$ is a solution of (1), then $(f\delta_1, \ldots, f\delta_r, fc, fd)$ is also a solution of (1) for any $f \in \mathbb{F}_q[t]$.

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Chang-Papanikolas-Yu criterion

Lemma (Chen)

(a) If $\delta_1, \cdots, \delta_r \in \overline{K}[t]$ satisfy equations of (1), then $\delta_1, \cdots, \delta_r \in \mathbb{F}_q[\theta, t]$.

(b) If $\delta_1, \cdots, \delta_r \in \mathbb{F}_q[\theta, t]$ satisfy equations of (1), then $\deg_\theta \delta_i \leq \frac{q(s_i+\cdots+s_r)}{q-1}$.

For simplicity, taking a twisting on both side of equations of (1), we have

$$\begin{aligned}
\delta_1^{(1)} &= \delta_1(t-\theta^q)^w + \delta_2 H_{s_1-1}(t-\theta^q)^w + dH_{w-1}(t-\theta^q)^w;\\
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The algorithm

Now, we are ready to write down an algorithm for determining whether any given MZV is zeta-like. For simplicity, we only consider $r = 2$. Give any $\mathfrak{s} = (s_1, s_2) \in \mathbb{N}^2$ with $q - 1 \nmid s_1$ and $q - 1 \mid s_2$, we list the essential steps as follows.

Step 1 : Compute the Anderson-Thakur polynomials $H_{s_1-1}, H_{s_2-1}, H_{w-1}$ and $a_{\mathfrak{s}'}$ where $\mathfrak{s}' = (s_2)$.

Step 2 : Let $n_2 = \frac{qs_2}{q-1}$ and write $\delta_2 = b_{2,n_2}\theta^{n_2} + \cdots + b_{2,0}$ with $b_{2,i}$ as variables. Expand the following equation

$$b_{2,n_2}\theta^{qn_2} + \cdots + b_{2,0} = (b_{2,n_2}\theta^{n_2} + \cdots + b_{2,0})(t - \theta^q)^{s_2} + a_{\mathfrak{s}'} H_{s_2-1}(t - \theta^q)^{s_2}.$$

Compare the coefficients of θ^i, and we can solve $b_{2,i} \in \mathbb{F}_q[t]$ for $i = 0, \ldots, n_2$ since $\zeta_A(s_2)$ is Eulerian.

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Step 1 : Compute the Anderson-Thakur polynomials $H_{s_1-1}, H_{s_2-1}, H_{w-1}$ and $a_{\mathfrak{s}'}$ where $\mathfrak{s}' = (s_2)$.

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$$b_{2,n_2}\theta^{qn_2}+\cdots+b_{2,0} = (b_{2,n_2}\theta^{n_2}+\cdots+b_{2,0})(t-\theta^q)^{s_2}+a_{\mathfrak{s}'}H_{s_2-1}(t-\theta^q)^{s_2}.$$

Compare the coefficients of θ^i, and we can solve $b_{2,i} \in \mathbb{F}_q[t]$ for $i = 0, \ldots, n_2$ since $\zeta_A(s_2)$ is Eulerian.

The algorithm

Step 3 : Let $n_1 = \left[\frac{qw}{q-1}\right]$ and write $\delta_1 = b_{1,n_1}\theta^{n_1} + \cdots + b_{1,0}$ with $b_{1,i}$ as variables. Expand the following equation

$$\delta_1^{(1)} = \delta_1(t-\theta^q)^w + \delta_2 H_{s_1-1}(t-\theta^q)^w \cdot f + H_{w-1}(t-\theta^q)^w \cdot d,$$

where $f, d \in \mathbb{F}_q[t]$ are also variables. Compare the coefficients of θ^i and write down these relations as follows

$$M_1 \begin{pmatrix} b_{1,n_1} \\ b_{1,n_1-1} \\ \vdots \\ b_{1,0} \\ f \\ d \end{pmatrix} = 0,$$

where M_1 is a matrix of entries in $\mathbb{F}_q[t]$. Compute the rank of M_1 over $\mathbb{F}_q(t)$. If the rank of M_1 is equal to n_1+3, then $\zeta_A(s_1,\ldots,s_r)$ is non zeta-like. Otherwise, $\zeta_A(s_1,\ldots,s_r)$ is zeta-like.

The algorithm

Example : When $q = 3$ and $\mathfrak{s} = (1,2)$, we have $c = t^3 - t$ and $\delta_2 = \theta^3 + 2t^3$. Note that $n_1 = \left[\frac{9}{2}\right] = 4$ and $H_0, H_2 = 1$. Write $\delta_1 = b_4\theta^4 + b_3\theta^3 + b_2\theta^2 + b_1\theta + b_0$ and expand the following equation

$$\delta_1^{(1)} = \delta_1(t-\theta^3)^3 + (\theta^3 + 2t^3)(t-\theta^3)^3 \cdot f + (t-\theta^3)^3 \cdot d.$$

Compare the coefficients of θ^i for $i = 0, 1, \ldots, 13$, we have

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & t^3 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ t^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^3 & 0 & -1 & 0 & t^3 & 0 \\ 0 & 0 & t^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^3-1 & -t^6 & t^3 \end{pmatrix}.$$

Then the rank of M is equal to $6 < n_1 + 3 = 7$, so $\zeta_A(1,2)$ is zeta-like.

Observations and conjectures

Basing on our list of zeta-like tuples, we and J. Yu have made the following conjectures on specific families of MZV's:

Conjecture 1

For any q, and any $m \geq 0$ such that $1 \leq p^m \leq q$ and $r \geq 2$, the MZV

$$\zeta_A\left(1, p^m(q-1), p^m q(q-1), \ldots, p^m q^{r-2}(q-1)\right)$$

is zeta-like.

Remark : Conjecture 1 together with exact formulas for these zeta-like MZV's was proved by Lara Rodríguez-Thakur in the case $m = 0$. For general m, it was proved by Chen.

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Conjecture 2

For any q, we have

(a) For any $n > 0$, $r \geq 2$ and $m \geq 1$ such that $p^m \leq q$, the MZV

$$\zeta_A(q^n - \sum_{i=1}^{s} q^{k_i}, p^m q^{n-1}(q-1), \ldots, p^m q^{n+r-3}(q-1))$$

is zeta-like, where $1 \leq s \leq q-1$, $0 \leq k_i < n$ such that $(q-1)s_1 \leq s_2$.

(b) In the case of depth $r = 3$, the following identity should hold

$$\zeta_A(1, q(q-1), q^3 - q^2 + q - 1) = \frac{[3] - 1}{[3][2][1]^{q^2-q+1}} \zeta_A(q^3),$$

where $[n] = t^{q^n} - t$ for $n \in \mathbb{N}$.

Observations and conjectures

Remark : Conjecture 1 is a special case of Conjecture 2 (a). Conjecture 2(a) and (b) have been proved also by Chen.

Remark : Lara Rodríguez-Thakur conjectured that if $\zeta_A(s_1, \ldots, s_r)$ is zeta-like than $\zeta_A(s_1, s_2)$ must be zeta-like. Note that CPY proved that $\zeta_A(s_2, \ldots, s_r)$ must be Eulerian. In other words, there is the "slicing" phenomenon : zeta-like $(s_1, \ldots, s_r)$ sliced into depth 2 zeta-like (s_1, s_2) and depth $r-1$ Eulerian $(s_2, \ldots, s_r)$.

In Conjecture 2 (a), the depth r zeta-like comes from combining the depth 2 zeta-like $\left(q^n - \sum_i q^{k_i}, p^m q^{n-1}(q-1)\right)$ and the non-primitive canonical Eulerian $p^m\left(q^{n-1}(q-1), \ldots, q^{n+r-3}(q-1)\right)$ in CPY's paper.

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Observations and conjectures

The conjectured list given in CPY's paper says there are very few infinite families of primitive Eulerian MZV's for given q. In fact, when $q > 2$, only exists the canonical family

$$\left(q^{\ell} - 1, q^{\ell}(q-1), q^{\ell+1}(q-1), \ldots, q^{\ell+r-2}(q-1)\right)$$

for depth $r \geq 3$. The families of zeta-like MZV's given in our conjectures may well exhaust all zeta-like MZV's with depth $r > 2$. Depth 2 zeta-like MZV's are abundant, and still awaits more explanations.

Observations and conjectures

Put $\ell_0 = 1$ and $\ell_n = \prod_{i=1}^{n}(t - t^{q^i})$ for $n \in \mathbb{N}$.

Theorem (Chen-K.)

(a) Let $n \geq 0$ and $a_i, b_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, \ldots, n$. Assume $a_n + b_n \geq q$, $\sum_{i=0}^{n} a_i \leq q-1$ and $1 \leq \sum_{i=0}^{n} b_i \leq q$. For $s_1 = q^{n+1} - \sum_{i=0}^{n} a_i q^i$ and $s_2 = \sum_{i=0}^{n} b_i(q^{n+1} - q^i)$, we have

$$\zeta_A(s_1, s_2) = \ell_{n+1}^{a_0} \ell_1^{(q^n a_n - q^{n-1} a_{n-1} - q^{n+1})} \prod_{i=1}^{n-1} \ell_{n-i+1}^{(q^i a_i - q^{i-1} a_{i-1})} \zeta_A(s_1 + s_2).$$

(b) For $1 \leq a \leq q$, we have

$$\zeta_A(aq - a + 1, q^3 - aq + a - 1) = \frac{1 - [2]^q}{\ell_1^{(a-1)(q+1)} \ell_2^{(q-a+1)}} \zeta_A(q^3).$$

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Remark : From our data, when $q = p$, we find that the above theorem and Lara Rodríguez-Thakur's theorem include all depth 2 zeta-like tuples. We may well exhaust all zeta-like MZV's with depth 2 when $q = p$.

For $q \neq p$, form our data, we conjecture that

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Thanks

Thank you for your attention!!