Degeneration of Calabi-Yau 3-folds and 3-forms

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Overview

Work based on arXiv: 2405.04827 and ongoing projects.

1. Motivations
2. Set-up
3. Degeneration of 3-forms
4. Integrability of 3-forms
5. Nilpotent Jordan-Calabi-Yau geometry

SYZ conjecture

The Strominger-Yau-Zaslow (SYZ) conjecture:

- Near large complex structure limits, Calabi-Yau manifolds admit special Lagrangian torus fibrations with singularities
- The fibration is semi-flat and the base admits an affine structure with singularities, together with a Hessian metric solving the real Monge-Ampère equation
- Mirror manifold can be constructed by taking the dual torus fibration with quantum corrections

Lots of aspects

- Gross-Siebert program
- Gross-Wilson, Joyce, Collins-Jacob-Lin, Lau-Lee-Lin, Y. Li et al.

Some questions

Questions

- Why Lagrangian fibrations/foliations arise naturally from degeneration of CY structures?
- Semi-flatness?
- Real Monge-Ampère equation?
- ...

Goal today:
Present a direct explanation to some of the ingredients in questions above.

For simplicity, we work with complex dimension 3, though many results generalize to higher dimensions.

Set-up

Usual set-up: consider a polarized degeneration of CY3's
Our set-up: fix a symplectic manifold (M, ω) and consider a family of CY-structures on (M, ω).

How to pass from usual set-up to ours?

- All smooth fibers are diffeomorphic, hence all can be identified with M
- The symplectic form have the same de Rham class, can all be pullback to ω by Moser's trick

Comparison

- Disadvantage: lose track of monodromy
- Advantage: easier to take limits

Set-up cont'd

Our set-up:
Fix a symplectic manifold (M, ω), consider a family of CY-structures $\Omega_t = \varphi_t + i\hat{\varphi}_t$ on (M, ω)

Facts:

- J_t is determined by Ω_t
- For CY3, Ω_t is determined by φ_t via Hitchin's construction
- φ_t is a "positive" primitive 3-form
- We may assume that Ω_t/φ_t is Ricci-flat by running the Type IIA flow, meaning $\mathrm{d}\varphi_t = \mathrm{d}\hat{\varphi}_t = 0$, $|\varphi_t|^2 = \mathrm{const.}$

Q: Is there a limit $\lim_{t\to 0} \varphi_t$?

Set-up cont'd

Our set-up:

Let's assume that $\lim_{t\to 0}\varphi_t = \varphi_0$ smoothly away from some singular locus. φ_0 should be a degenerate primitive 3-form satisfying strong integrability conditions.

Extra freedom: one may rescale Ω_t by suitable constants to help φ_t converge.
Constraint: want φ_t to converge in a specific way with rich geometric information on $(M^\circ, \omega, \varphi_0)$.

Need to understand the degeneration of (primitive) 3-forms.

Classification of 3-forms

Let V be a 6D vector space over $\mathbb{R}$, $\varphi \in \wedge^3 V^*$.
The classification of $\mathrm{GL}(V)$-orbits in $\wedge^3 V^*$ has been known for long time (Reichel 1907', Gurevich 35', Chan 98', Bryant 06',...)

Orbit	Dimesion	Normal Form
$\mathcal{O}_-$	20	$e^{135} - e^{146} - e^{236} - e^{245}$
$\mathcal{O}_+$	20	$e^{123} + e^{456}$
$\mathcal{O}_0$	19	$e^{146} + e^{236} + e^{245}$
$\mathcal{O}_1$	15	$e^{135} + e^{245}$
$\mathcal{O}_3$	10	e^{135}
$\mathcal{O}_6 = \{0\}$	0	0

$\mathcal{O}_- \rightsquigarrow$ complex geometry: Hitchin 00', Donaldson-Lehmann 24'...
$\mathcal{O}_+ \rightsquigarrow$ paracomplex geometry
other orbits $\rightsquigarrow$???

Equivariant polynomials

We introduce a few equivariant homogeneous polynomials on the space of 3-forms.

Name	Degree	Value	Definition
Id	1	$\wedge^3 V^*$	$\mathrm{Id}(\varphi) = \varphi$
K	2	$\mathrm{End}\ V \otimes \wedge^6 V^*$	$K(\varphi)(v) = -\iota_v\varphi \wedge \varphi$
F	3	$\wedge^3 V^* \otimes \wedge^6 V^*$	$\iota_v F(\varphi) = -2\iota_{K(\varphi)(v)}\varphi$
Q	4	$(\wedge^6 V^*)^{\otimes 2}$	$Q(\varphi) = -\varphi \wedge F(\varphi)$

Facts:

- $\mathcal{O}_- = \{Q < 0\}$, $\mathcal{O}_+ = \{Q > 0\}$
- $\mathcal{O}_0 \coprod \mathcal{O}_1 \coprod \mathcal{O}_3 \coprod \mathcal{O}_6 = \{Q = 0\}$
- $\mathcal{O}_1 \coprod \mathcal{O}_3 \coprod \mathcal{O}_6 = \{F = 0\}$
- $\mathcal{O}_3 \coprod \mathcal{O}_6 = \{K = 0\}$
- $\mathcal{O}_6 = \{\mathrm{Id} = 0\}$

Hitchin's construction

Identity: $K^2(\varphi) = \frac{\mathrm{Id}_V}{4} Q(\varphi)$.
If $Q(\varphi) < 0$, one can define a complex structure

$$J(\varphi) = \frac{2K(\varphi)}{\sqrt{-Q(\varphi)}} : V \to V$$

and let $\hat{\varphi} = J(\varphi)^* \varphi$. It turns out that $\Omega(\varphi) = \varphi + i\hat{\varphi}$ is a nonzero $(3,0)$-form with respect to $J(\varphi)$.
For $\varphi \in \mathcal{O}_-$, we have

- $K(\varphi) = \frac{\sqrt{-Q(\varphi)}}{2} \cdot J(\varphi) \approx \frac{|\varphi|^2}{2} J(\varphi)$
- $F(\varphi) = \sqrt{-Q(\varphi)} \cdot \hat{\varphi} \approx |\varphi|^2 \hat{\varphi}$
- $Q(\varphi) \approx -|\varphi|^4$

Classification of primitive 3-forms

Consider a 6D symplectic space (V, ω) and a primitive 3-form $\varphi \in \wedge_0^3 V^*$. Similar classification of $\mathrm{Sp}(V, \omega)$-orbits in $\wedge_0^3 V^*$: Lychagin-Rubtsov 83', Banos 03', Bryant 06',...

Orbit	Dimension	Normal Form
$\mathcal{O}_-^{\pm}(\mu)$	13	$\mu(e^{135} - e^{146} \mp e^{236} \mp e^{245})$
$\mathcal{O}_+(\mu)$	13	$\mu(e^{135} + e^{246})$
$\mathcal{O}_0^{\pm}$	13	$e^{146} \pm e^{236} \pm e^{245}$
$\mathcal{O}_1^{\pm}$	10	$(e^{13} \mp e^{24}) \wedge e^5$
$\mathcal{O}_3 \cap \bigwedge_0^3 V^*$	7	e^{135}
$\mathcal{O}_6 = \{0\}$	0	0

Using the canonical volume form $\frac{\omega^3}{3!}$, we can think of $K(\varphi)$, $F(\varphi)$ and $Q(\varphi)$ as an endomorphism of V, a 3-form on V, and a scalar respectively.

Assumption

TFAE

- φ is the real part of a complex volume form
- φ is a "positive" primitive 3-form
- $\varphi \in \mathcal{O}^+_-(\mu)$ for some $\mu > 0$

Our set-up: on (M, ω), consider a family of Calabi-Yau structures determined by φ_t (real part of Ricci-flat holomorphic volume forms)

Key assumption: On an open (dense) subset M°, we have

$$\lim_{t \to 0} \varphi_t = \varphi_0$$

smoothly and φ_0 belongs to the orbit $\mathcal{O}^+_0$ pointwise

Goal: Study the geometry of $(M^\circ, \omega, \varphi_0)$

Examples

Ex. 0:

- $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \cong (\mathbb{R}^2/\mathbb{Z}^2, J_\tau)$
- J_τ determined by holomorphic $(1,0)$-form $\mathrm{d}x + \tau\mathrm{d}y$
- standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y$

Ex. 1:

- $(M,\omega) = E_\tau \times E_\tau \times E_\tau$ with product symplectic and complex structures
- $\Omega_\tau = \dfrac{1}{\tau^2}(\mathrm{d}x^1 + \tau\mathrm{d}y^1) \wedge (\mathrm{d}x^2 + \tau\mathrm{d}y^2) \wedge (\mathrm{d}x^3 + \tau\mathrm{d}y^3)$
- $\varphi_t = \mathrm{Re}\ \Omega_t$ where $\tau = \dfrac{i}{t}$ with $t > 0$
- $\varphi_0 = \lim_{t\to 0^+} \varphi_t = \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 + \mathrm{d}y^2 \wedge \mathrm{d}y^3 \wedge \mathrm{d}x^1 + \mathrm{d}y^3 \wedge \mathrm{d}y^1 \wedge \mathrm{d}x^2$
- (M, ω, φ_0) has "flat" geometry

Examples cont'd

Ex. 2:

- $\pi : X \to \mathbb{P}^1$ elliptic fibered K3 surface with complex structure I
- hyperkähler triple ω_I, ω_J and ω_K
- holomorphic volume form $\Omega_I = \omega_J + i\omega_K$
- ω_t is the Ricci-flat Kähler metric in the class $t\omega_I + \pi^*\omega_{FS}$
- $\int_X \omega_I \wedge \pi^*\omega_{FS} = c \int_X \omega_I^2$
- HK rotation: symplectic manifold (X, ω_K) with a family of CY structures $\dfrac{\omega_t}{\sqrt{2ct+t^2}} + i\omega_J$
- $M = X \times (\mathbb{R}^2/\mathbb{Z}^2)$ with product symplectic structure $\omega_K + \omega_0$
- holomorphic volume form $\Omega_t = (\sqrt{2ct}\mathrm{d}x + i\mathrm{d}y) \wedge (\dfrac{\omega_t}{\sqrt{2ct+t^2}} + i\omega_J)$
- $\lim_{t\to 0} \varphi_t = \mathrm{d}x \wedge \lim_{t\to 0} \omega_t - \mathrm{d}y \wedge \omega_J$
- Tosatti 10': $\lim_{t\to 0} \omega_t = \pi^*\omega$ away from singular fibers

Various integrability

Let φ be a primitive 3-form on (M, ω). There are various kinds of integrabilities:

- integrable: $\mathrm{d}\varphi = 0$
- K-integrable: Nijenhuis tensor of $K(\varphi)$ vanishes
- F-integrable: $\mathrm{d}F(\varphi) = 0$
- Q-integrable: $\mathrm{d}Q(\varphi) = 0$, namely $Q(\varphi)$ is a constant

Theorem

Being integrable and F-integrable implies all other kinds of integrability conditions.

Definition

On a symplectic 6-manifold, a primitive 3-form is called *F-harmonic* if it is both integrable and F-integrable.

Ricci-flat condition

Fact: For a Calabi-Yau structure φ on (M, ω), being F-harmonic is equivalent to being Ricci-flat.

Recall key assumption: On an open (dense) subset M°, we have

$$\lim_{t \to 0} \varphi_t = \varphi_0$$

smoothly and φ_0 belongs to the orbit $\mathcal{O}_0^+$ pointwise.

The F-harmonicity of φ_t implies the F-harmonicity of φ_0.

Goal: study the geometry of $(M^\circ, \omega, \varphi_0)$, where φ_0 belongs to the orbit $\mathcal{O}_0^+$ pointwise and is F-harmonic $\rightsquigarrow$ nilpotent Jordan-Calabi-Yau geometry

Nilpotent Jordan-Calabi-Yau geometry

Let (M°, ω) be a symplectic 6-manifold, φ_0 an F-harmonic primitive 3-form on M° belonging to the orbit $\mathcal{O}_0^+$ pointwise. There are very rich geometric structures attached to the triple $(M^\circ, \omega, \varphi_0)$.

Theorem

$K = K(\varphi)$ is an endomorphism of TM° satisfying

- $\ker K = \mathrm{im}\ K$ is a Lagrangian subbundle of TM°.
- The Nijenhuis tensor of K vanishes: $K([KX, Y] + [X, KY]) - [KX, KY] = 0$.
- $\mathcal{L} := \ker K$ is a Lagrangian foliation

Conclusion: Lagrangian foliation (with singularities) arises naturally from certain degeneration of CY3-structures.

Nilpotent Jordan-Calabi-Yau geometry cont'd

- The symmetric bilinear form $g(\cdot,\cdot) = \omega(\cdot, K\cdot)$ induces a Riemannian metric on $TM^\circ/\mathcal{L} \cong \mathcal{L}$, hence a metric g on every leaf of $\mathcal{L}$.
- K defines a partial connection D on $\mathcal{L}$: $D_X Y = K[X, K^{-1}Y]$ for any X, Y tangent to $\mathcal{L}$.
- D is torsion-free and flat, defining an affine structure on leaves of $\mathcal{L}$.
- As leaves of a Lagrangian foliation $\mathcal{L}$, they carry a canonical affine structure characterized by the Bott connection ∇^B.
- g is a Hessian metric with respect to the affine structures D and ∇^B.
- D and ∇^B are dual to each other, related by Legendre transform.
- Leaves of $\mathcal{L}$ are canonically oriented, and the volume form of g is parallel under both D and ∇^B, namely in either affine coordinates, the potential of g solves a real MA equation with constant RHS.
- On each leaf, g has nonnegative Ricci curvature.
- If a leaf is compact, it must be a flat torus.

What are missing?

Open Questions:

- How do we know if $\lim_{t\to 0}\varphi_t$ exists (away from some locus)?
- When do we know the Lagrangian foliation is a fibration?
- How to produce an affine structure and a metric on the base if we have a fibration?
- How to distinguish two kinds of singularities: the points where $\lim_{t\to 0}\varphi_t$ does not exist and the points where $\lim_{t\to 0}\varphi_t$ degenerates further?
- More examples?
- ...

Let me know if you have any comments!

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