Introduction to generalized Kähler geometry

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1 Introduction

Generalized Kähler geometry was discovered by physicists Gates, Hull, and Roček in 1984 [12], when they sought to generalize the earlier work of Zumino, who showed that a Kähler structure on the target of the 2-dimensional sigma model endows the model with an action by the $N = (2, 2)$ supersymmetry algebra. They observed that the same occurs if the target is endowed with a generalized Kähler structure.

The literature on generalized Kähler geometry may be very roughly summarized as follows:

1. The (very large) physics literature, of which some key examples are [12, 27, 25], and most recently [24].
2. Relevant literature from complex geometry, especially [2].
3. The link to Hitchin's generalized geometry [22, 16, 19, 20]
4. The construction of many examples [23, 18], for example by generalized Kähler reduction [7, 8].
5. Hodge theory for generalized Kähler structures [17, 9, 3]
6. T-duality and generalized geometry [1]
7. Deformation theory of Generalized Kähler metrics [13, 21],
8. Curvature of generalized Kähler manifolds, and Kobayashi-Hitchin correspondence for vector bundles over generalized Kähler manifolds [15, 14]
9. Generalized Ricci Flow [11], which includes generalized Kähler–Ricci flow, a variant of pluriclosed flow.
10. Generalized Kähler geometry and symplectic groupoids, the generalized Kähler potential [5, 31]

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2 Generalized geometry

In generalized geometry, instead of using the tangent bundle TM of the manifold M to model geometric structures, we use an extension of this bundle. The only example we shall study in this course is

$$\mathbb{T}M = TM \oplus T^*M,$$

which is useful for understanding T-duality, Mirror symmetry, and Type I and II string theories. There is also

$$TM \oplus \mathfrak{g}_M \oplus T^*M,$$

where $\mathfrak{g}_M$ is the adjoint bundle of a principal G-bundle, which is useful for understanding the Hull–Strominger system and Heterotic string theory. See these references: [4, 10, 29]. There are even more complicated examples, useful for capturing other supergravity theories, see for example [6] and its references. All of the above are examples of *Courant algebroids*, introduced in [26]. Very generally, the Courant algebroid is supposed to be part of the background or substrate, on top of which the geometry is defined. Once we have introduced the simplest kind of Courant algebroid, we will explore several geometric structures on it, including:

1. Dirac structure
2. Generalized metric
3. Generalized complex structure

The first is a generalization of a foliation, whereas the second and third are generalizations of Riemannian metrics and complex structures, respectively.

2.1 The Courant algebroid $\mathbb{T}M$

Split signature metric and spinors

A section $X + \xi \in C^\infty(\mathbb{T}M)$ acts on a differential form $\rho \in \Omega(M)$ via interior and exterior product:

$$(X + \xi) \cdot \rho = i_X\rho + \xi \wedge \rho.$$

If we square this action, we obtain

$$(X + \xi) \cdot ((X + \xi) \cdot \rho) = i_X(\xi \wedge \rho) + \xi \wedge i_X\rho = (i_X(\xi))\rho,$$

so that if we define a symmetric bilinear form on the bundle $\mathbb{T}M$ as follows:

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(i_X\eta + i_Y\xi),$$

we obtain a metric of signature (n, n) for $n = \dim M$ on $\mathbb{T}M$ such that the squared action satisfies

$$(X + \xi) \cdot ((X + \xi) \cdot \rho) = \langle X + \xi, X + \xi \rangle \rho.$$

This implies that the Clifford algebra bundle $Cl(\mathbb{T}M, \langle \cdot, \cdot \rangle)$ acts on the bundle $\wedge^\bullet T^*M$ of differential forms. This representation is actually the spin representation of the real Clifford algebra of signature (n, n). This representation is irreducible for the action of the Clifford algebra, but if we consider the Spin subgroup $Spin(n, n)$, then this decomposes into a sum of irreducibles: the *even* and *odd* spinors, corresponding to differential forms of even and odd degree, respectively:

$$S = \wedge^\bullet T^*M = \wedge^{ev} T^* \oplus \wedge^{od} T^* = S^+ \oplus S^-.$$

In conclusion, on any manifold M, the natural bundle $\mathbb{T}M$ is endowed with a metric of split signature, and we may view the differential forms of M as its spinors.

Exercise 2.1. Prove that the bundle of Lie algebras $\mathfrak{so}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$ is naturally isomorphic to

$$\wedge^2 TM \oplus \mathsf{End}(TM) \oplus \wedge^2 T^*M.$$

Show that any section $\beta + A + B$ of the above defines the following block endomorphism of $\mathbb{T}M$:

$$\begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $B \in \wedge^2 T^*$, for example, determines the transformation $B : X + \xi \mapsto i_X B$. Use this to compute the Lie bracket. □

Exercise 2.2. The Lie algebra $\mathfrak{so}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$ acts on spinors, and in particular, $B \in \Omega^2(M)$ acts on $\rho \in \Omega^\bullet(M)$ via $\rho \mapsto B \wedge \rho$. This action of $\Omega^2(M)$ is called a *B-field transformation* in physics. Prove that the actions intertwine, i.e. show that

$$-B \wedge ((X + \xi) \cdot \rho) = (B(X + \xi)) \cdot \rho + (X + \xi) \cdot (-B \wedge \rho).$$

By exponentiating this action, show that

$$e^{-B}((X + \xi) \cdot \alpha) = (e^B(X + \xi)) \cdot e^{-B} \alpha.$$

In other words, if $c(X + \xi) = (X + \xi)\cdot$ is the operator of Clifford action by $X + \xi$, then the above equation may be interpreted as follows:

$$c(e^B(X + \xi)) = e^{-B} \circ c(X + \xi) \circ e^B. \tag{1}$$

□

The spin representation has a natural bilinear form called the Chevalley pairing: for $\alpha, \beta \in \Omega^\bullet(M)$, their pairing is the top degree form

$$\langle \alpha, \beta \rangle_S = (\alpha \wedge \beta^\top)_{\mathrm{top}},$$

where $(\rho)_{\mathrm{top}}$ denotes the component of degree $\dim M$ of ρ, and $\beta \mapsto \beta^\top$ is the reversal anti-automorphism of the differential forms, i.e.

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})^\top = dx^{i_k} \wedge \cdots \wedge dx^{i_1}.$$

In other words, $\beta^\top = (-1)^{k(k-1)/2}\beta$ for β of degree k.

Exercise 2.3. Show the identity, for all $\alpha, \beta \in \Omega^\bullet(M)$ and $X + \xi \in \mathbb{T}M$,

$$\langle (X+\xi)\cdot\alpha, (X+\xi)\cdot\beta \rangle_S = \langle X+\xi, X+\xi \rangle \langle \alpha, \beta \rangle_S,$$

and conclude that the Chevalley pairing is invariant under the action of the identity component of $Spin(\mathbb{T}M, \langle \cdot, \cdot \rangle)$. In particular, check explicitly that for $B \in C^\infty(\wedge^2 T^*M)$,

$$\langle e^B \alpha, e^B \beta \rangle_S = \langle \alpha, \beta \rangle_S.$$

□

Exercise 2.4. Show the identity

$$\langle \alpha, \beta \rangle_S = (-1)^{n(n-1)/2} \langle \beta, \alpha \rangle_S.$$

Write the Chevalley pairing explicitly in the following cases:

1. Even or odd forms on a 4-manifold,
2. Even or odd forms on a 3-manifold,
3. Even or odd forms on a 2-manifold.

Verify that the Chevalley pairing is symmetric in the first case, and skew-symmetric in the second and third cases. □

The Courant bracket

The Lie bracket of vector fields is dual to the de Rham exterior derivative, in a sense made precise by the following identity, for all vector fields X, Y and differential forms ρ:

$$i_{[X,Y]}\rho = [[d, i_X], i_Y]\rho.$$

In view of our earlier discussion of the action of $\mathbb{T}M$ on forms, we may extend the Lie bracket to a *Courant* bracket, as follows. Recall that $c(X+\xi) = (X+\xi)\cdot$ denotes the Clifford action of a section of $\mathbb{T}M$.

$$c([X+\xi, Y+\eta]) = [[d, c(X+\xi)], c(Y+\eta)]. \tag{2}$$

Exercise 2.5. With the above definition, prove that

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - i_Y d\xi.$$

$\square$

Note that this bracket is not skew-symmetric, and satisfies

$$[X+\xi, X+\xi] = d\langle X+\xi, X+\xi\rangle,$$

or, more generally,

$$[X+\xi, Y+\eta] + [Y+\eta, X+\xi] = 2d\langle X+\xi, Y+\eta\rangle.$$

In particular, if sections are taken from an isotropic subbundle of $\mathbb{T}M$, the bracket will be skew-symmetric.

In fact, the Courant bracket satisfies a version of the Jacobi identity, as follows.

Exercise 2.6. Using the definition (2), prove the Jacobi identity for the Courant bracket, i.e.

$$[[X+\xi, Y+\eta], Z+\zeta] = [X+\xi, [Y+\eta, Z+\zeta]] - [Y+\eta, [X+\xi, Z+\zeta]].$$

$\square$

Remark 2.7. The Courant bracket is close to being a Lie algebra. In fact, as shown in [28], it defines a L_∞ algebra structure on the following complex, concentrated in degrees -1, 0:

$$C^\infty(M,\mathbb{R}) \xrightarrow{d} C^\infty(\mathbb{T}M)\ .$$

The binary bracket (of degree zero) vanishes in degree -1 and is the skew-symmetrization of the Courant bracket in degree 0. Between degrees -1 and 0, the bracket is $[X+\xi, f] = \frac{1}{2}X(f)$, and finally the ternary bracket (of degree -1) has only one component, namely

$$[X+\xi, Y+\eta, Z+\zeta] = \tfrac{1}{3}(\langle[X+\xi, Y+\eta], Z+\zeta\rangle + c.p.).$$

$\square$

It is natural to ask whether any part of the Lie algebra $\mathfrak{so}(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ acts in such a way as to preserve the Courant bracket. Focusing on B-field transformations, we obtain the following identity: let $U = X+\xi$, $V = Y+\eta$ and $B \in \Omega^2(M)$. Then from the following identity:

$$[[d, e^{-B}c(U)e^{B}], e^{-B}c(V)e^{B}] = e^{-B}[[e^{B}de^{-B}, c(U)], c(V)]e^{B},$$

and using the fact that

$$e^{B}de^{-B} = d - dB\wedge\cdot,$$

we conclude that under the condition that B is closed, the B-field transformation e^B is a symmetry of the Courant bracket, namely

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B[X+\xi, Y+\eta].$$

Exercise 2.8. Fill in the details in the above argument. Also, try to prove the fact that closed B-field transformations are the only sections of $\mathfrak{so}(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ that preserve the Courant bracket. See [16] for a proof. □

In the above argument, we saw that if $B \in \Omega^2(M)$ is *not closed*, then it does not preserve the Courant bracket; instead it takes the Courant bracket to a *twisted* Courant bracket, that is,

$$[e^B(U), e^B(V)] = e^B[U,V]_{dB},$$

where the bracket on the right hand side is defined in the same way as the Courant bracket, but for the differential

$$(d - dB\wedge\cdot) : \Omega^{ev/od}(M) \to \Omega^{od/ev}(M).$$

Since exact 3-forms and closed 3-forms are locally equivalent, we may define, for any closed 3-form H, a *twisted de Rham* operator

$$d_H = (d - H\wedge\cdot) : \Omega^{ev/od}(M) \to \Omega^{od/ev}(M),$$

and a *twisted* Courant bracket

$$c([X+\xi, Y+\eta]_H) = [[d_H, c(X+\xi)]c(Y+\eta)].$$

Exercise 2.9. Prove that the twisted Courant bracket is given by

$$[X+\xi, Y+\eta]_H = [X,Y] + L_X\eta - i_Y d\xi + i_Y i_X H.$$

□

In the study of 2-dimensional sigma models in physics, the closed 3-form H is known as the Wess–Zumino term, and in string theory it is known as the Neveu–Schwarz 3-form flux. In these theories, it is important that H has integral periods, and in fact it should be viewed as the curvature of a $U(1)$ gerbe with connection and curving.

Given a manifold M equipped with a closed 3-form H, the tuple $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ is known as an *exact Courant algebroid*. The action of B-field transformations provides isomorphisms

$$[e^B(U), e^B(V)]_H = e^B[U,V]_{H+dB}$$

between Courant algebroids whose 3-forms are cohomologous. Indeed, as shown in [30], exact Courant algebroids are classified by the cohomology class $[H] \in H^3(M,\mathbb{R})$, known as the Ševera class.

2.2 Generalized metrics

The structure group of the metric bundle $\mathbb{T}M$ is $O(n,n)$; a reduction to the maximal compact subgroup $O(n) \times O(n)$ is called a *generalized Riemannian metric*.

Definition 2.10. A *generalized Riemannian metric* is a maximal positive-definite subbundle

$$V_+ \subset \mathbb{T}M. \tag{3}$$

The orthogonal complement $V_- = V_+^\perp$ relative to the split-signature metric is then maximally negative-definite, and we have a decomposition, orthogonal with respect to $\langle \cdot, \cdot \rangle$, as follows:

$$\mathbb{T}M = V_+ \oplus V_-.$$

Since the subbundles TM, T^*M of $\mathbb{T}M$ are null, in fact maximal isotropic, it follows that the projection along T^*M, i.e.

$$\pi : \mathbb{T}M \to TM,$$

defines an isomorphism of bundles

$$V_\pm \xrightarrow[\pi]{\cong} TM \; .$$

Exercise 2.11. Prove that a generalized Riemannian metric is given by the graph of a general 2-tensor $g + b \in C^\infty(S^2T^* \oplus \wedge^2 T^*)$ whose symmetric part g is positive-definite. That is,

$$V_+ = \{X + g(X) + b(X) \ : \ X \in TM\}.$$

Conclude that the orthogonal complement is then given by

$$V_- = \{X - g(X) + b(X) \ : \ X \in TM\}.$$

□

Since a generalized Riemannian metric determines the decomposition (3), it can be described in terms of an operator $\mathbb{G} : \mathbb{T}M \to \mathbb{T}M$ as follows:

$$\mathbb{G} = 1|_{V_+} + (-1)|_{V_-}.$$

Exercise 2.12. If the generalized Riemannian metric is given by $g+b$ as above, show that the corresponding operator is, in block form,

$$\mathbb{G} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}.$$

□

Assume we are on an oriented manifold, and choose an oriented orthonormal basis $(e_1, \ldots, e_n)$ for V_+ over any point. Then the product

$$* = e_n \cdots e_1 \in Cl(V_+) \subset Cl(\mathbb{T}M)$$

is a well-defined element of the Clifford algebra, independent of the chosen basis, and is therefore a global section of the Clifford algebra bundle. We call this element the *generalized Hodge star*. It acts on differential forms via the spin representation.

Exercise 2.13. Prove that the classical Hodge star $\star$ can be obtained from $*$ as follows. Assume $b = 0$, so that $V_+ = \mathrm{Gr}(g)$, for g a Riemannian metric. Then the Hodge star of g is

$$\star\rho = (*\rho)^\top.$$

□

Exercise 2.14. Prove the identity

$$*^2 = (-1)^{n(n-1)/2}.$$

□

2.3 Dirac structures

Fix a manifold M with closed 3-form H, and let $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ be the associated Courant algebroid. The first geometric structure we will consider is called a Dirac structure:

Definition 2.15. A Dirac structure is an involutive maximal isotropic subbundle $L \subset \mathbb{T}M$.

2.4 Generalized complex structures

3 Generalized Kähler geometry

3.1 Bi-Hermitian geometry and the pluriclosed condition

3.2 Generalized complex formulation

3.3 Generalized Hodge decomposition

3.4 Examples

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