Average values of higher moments of quadratic L-functions over rational function fields

Chih-Yun Chuang

Department of Mathematics, National Taiwan University, Taiwan

Taipei-Xian Number theory Workshop
Oct. 13, 2015

Outline

1. Average values of quadratic L-functions
   - Notations
   - Quadratic L-functions
   - The non-square case
   - The square-free case
   - Applications

2. The Proof
   - A type of quadratic Gauss sum
   - Rearrange our sums

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\mathrm{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\mathrm{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\text{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\mathrm{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\mathrm{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Notations

k : a rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$, where p is an odd prime.

A : the polynomial ring $\mathbb{F}_q[t]$.

∞ : the infinite place, which corresponds to degree valuation v_∞.

A^+ : the set of all monic polynomials.

P : a monic irreducible polynomial in A, which corresponds to a finite place of k.

$\text{sgn}(b)$: the leading coefficient of a polynomial $b \in A$.

Quadratic symbol in rational function fields

Definition

If P does not divide $a \in A$, let $\left[\frac{a}{P}\right]$ be the unique element of $\mathbb{F}_q^\times$ such that

$$a^{\frac{q^{\deg P}-1}{2}} \equiv \left[\frac{a}{P}\right] \mod P.$$

If $P \mid a$ define $\left[\frac{a}{P}\right] = 0$. The symbol $\left[\frac{a}{P}\right]$ called the quadratic residue symbol.

We extend the above definition of the quadratic symbol to the case where the prime P is replaced with an arbitrary non-zero element $b \in A$.

Quadratic symbol in rational function fields

Definition

If P does not divide $a \in A$, let $\left[\frac{a}{P}\right]$ be the unique element of $\mathbb{F}_q^\times$ such that

$$a^{\frac{q^{\deg P}-1}{2}} \equiv \left[\frac{a}{P}\right] \mod P.$$

If $P \mid a$ define $\left[\frac{a}{P}\right] = 0$. The symbol $\left[\frac{a}{P}\right]$ called the quadratic residue symbol.

We extend the above definition of the quadratic symbol to the case where the prime P is replaced with an arbitrary non-zero element $b \in A$.

Definition

Let $b \in A$, $b \neq 0$, and $b = \mathrm{sgn}(b) \cdot \prod_{i=1}^{N} P_i^{f_i}$ be the prime decomposition of b. If $a \in A$, define

$$\left[\frac{a}{b}\right] := \prod_{i=1}^{N} \left[\frac{a}{P_i}\right]^{f_i}.$$

The quadratic symbol satisfies the following quadratic reciprocity law:

Theorem

Let $a, b \in A$ be relatively prime, non-zero elements. Then,

$$\left[\frac{a}{b}\right]\left[\frac{b}{a}\right] = (-1)^{\frac{q-1}{2}\deg a \deg b}\,\mathrm{sgn}_2(a)^{\deg b}\,\mathrm{sgn}_2(b)^{-\deg a}.$$

Here $\mathrm{sgn}_2(a)$ is the leading coefficient of a raised to the $\frac{q-1}{2}$ power.

Quadratic symbol in rational function fields

Definition

Let $b \in A$, $b \neq 0$, and $b = \operatorname{sgn}(b) \cdot \prod_{i=1}^{N} P_i^{f_i}$ be the prime decomposition of b. If $a \in A$, define

$$\left[\frac{a}{b}\right] := \prod_{i=1}^{N} \left[\frac{a}{P_i}\right]^{f_i}.$$

The quadratic symbol satisfies the following quadratic reciprocity law:

Theorem

Let $a, b \in A$ be relatively prime, non-zero elements. Then,

$$\left[\frac{a}{b}\right]\left[\frac{b}{a}\right] = (-1)^{\frac{q-1}{2}\deg a \deg b} \operatorname{sgn}_2(a)^{\deg b} \operatorname{sgn}_2(b)^{-\deg a}.$$

Here $\operatorname{sgn}_2(a)$ is the leading coefficient of a raised to the $\frac{q-1}{2}$ power.

L-functions

Given $m \in A$ non-square, define the function $\chi_m(n) := \left[\frac{m}{n}\right]$ for $n \in A - \{0\}$. We are interested in the L-function associated to χ_m which is defined by

$$L(s, \chi_m) := \sum_{n \in A^+} \frac{\chi_m(n)}{q^{s \deg n}} = \prod_P (1 - \chi_m(P) q^{-s \deg P})^{-1},$$

$$\text{on } \Re(s) > 1.$$

In fact, $L(s, \chi_m)$ is a polynomial of degree at most $\deg m - 1$ in q^{-s}.

L-functions

Given $m \in A$ non-square, define the function $\chi_m(n) := \left[\frac{m}{n}\right]$ for $n \in A - \{0\}$. We are interested in the L-function associated to χ_m which is defined by

$$L(s, \chi_m) := \sum_{n \in A^+} \frac{\chi_m(n)}{q^{s \deg n}} = \prod_P (1 - \chi_m(P) q^{-s \deg P})^{-1},$$

$$\text{on } \Re(s) > 1.$$

In fact, $L(s, \chi_m)$ is a polynomial of degree at most $\deg m - 1$ in q^{-s}.

Suppose that $m \in A$ is square-free. We classify quadratic function fields $K := k(\sqrt{m})$ according to whether ∞ splits, is inert, or ramified in K/k. This is analogous to classifying quadratic number fields as real or imaginary. That is

- If $\deg m$ is even and $\operatorname{sgn}(m) \in (\mathbb{F}_q^\times)^2$, then ∞ splits in K/k.
- If $\deg m$ is even, and $\operatorname{sgn}(m) \notin (\mathbb{F}_q^\times)^2$, then ∞ is inert in K/k.
- If $\deg m$ is odd, then ∞ ramifies in K/k.

Let

$$\lambda_\infty(m) := \begin{cases} 0, & \text{if } \infty \text{ is ramified in } k(\sqrt{m})/k; \\ -1, & \text{if } \infty \text{ is inert in } k(\sqrt{m})/k; \\ 1, & \text{if } \infty \text{ splits in } k(\sqrt{m})/k. \end{cases}$$

Suppose that $m \in A$ is square-free. We classify quadratic function fields $K := k(\sqrt{m})$ according to whether ∞ splits, is inert, or ramified in K/k. This is analogous to classifying quadratic number fields as real or imaginary. That is

- If $\deg m$ is even and $\operatorname{sgn}(m) \in (\mathbb{F}_q^\times)^2$, then ∞ splits in K/k.
- If $\deg m$ is even, and $\operatorname{sgn}(m) \notin (\mathbb{F}_q^\times)^2$, then ∞ is inert in K/k.
- If $\deg m$ is odd, then ∞ ramifies in K/k.

Let

$$\lambda_\infty(m) := \begin{cases} 0, & \text{if } \infty \text{ is ramified in } k(\sqrt{m})/k; \\ -1, & \text{if } \infty \text{ is inert in } k(\sqrt{m})/k; \\ 1, & \text{if } \infty \text{ splits in } k(\sqrt{m})/k. \end{cases}$$

The complete L-function of $L(s, \chi_m)$ is given by

$$L^*(s, \chi_m) := (1 - \lambda_\infty(m) q^{-s})^{-1} \cdot L(s, \chi_m).$$

Let $\mathfrak{m}(\chi_m) := \deg m - 1 - |\lambda_\infty(m)|$. Then the functional equation for the complete L-function $L^*(s, \chi_m)$ is as follows:

$$L^*(s, \chi_m) = q^{\mathfrak{m}(\chi_m)(1/2 - s)} L^*(1 - s, \chi_m),$$

which implies that $L(s, \chi_m)$ is a polynomial of degree $\deg m - 1$ in q^{-s}.

The complete L-function of $L(s, \chi_m)$ is given by

$$L^*(s, \chi_m) := (1 - \lambda_\infty(m) q^{-s})^{-1} \cdot L(s, \chi_m).$$

Let $\mathfrak{m}(\chi_m) := \deg m - 1 - |\lambda_\infty(m)|$. Then the functional equation for the complete L-function $L^*(s, \chi_m)$ is as follows:

$$L^*(s, \chi_m) = q^{\mathfrak{m}(\chi_m)(1/2 - s)} L^*(1 - s, \chi_m),$$

which implies that $L(s, \chi_m)$ is a polynomial of degree $\deg m - 1$ in q^{-s}.

L-functions

Let B be the integral closure of A in K/k. Let ζ_A (reps. ζ_B) be the zeta function of A (reps. B):

- $\zeta_A(s) := \sum_{I \subset A} \mathbf{N}(I)^{-s} = \prod_{P} (1 - q^{-s \deg P})^{-1}$ on $\Re(s) > 1$, where $\mathbf{N}(I)$ denotes the absolute norm.
- $\zeta_B(s) := \sum_{I \subset B} \mathbf{N}(I)^{-s} = \prod_{\mathfrak{P}} (1 - \mathbf{N}(\mathfrak{P})^{-s})^{-1}$ on $\Re(s) > 1$, where the sum is over all non-zero ideals in B, and the product is over all non-zero prime ideals in B.

These zeta functions both have a simple pole at $s = 1$, and are rational functions in q^{-s}.

Then the L-function $L(s, \chi_m)$ satisfies

$$\zeta_B(s) = \zeta_A(s) \cdot L(s, \chi_m).$$

Let B be the integral closure of A in K/k. Let ζ_A (reps. ζ_B) be the zeta function of A (reps. B):

- $\zeta_A(s) := \sum_{I \subset A} \mathbf{N}(I)^{-s} = \prod_{P}(1 - q^{-s \deg P})^{-1}$ on $\Re(s) > 1$, where $\mathbf{N}(I)$ denotes the absolute norm.
- $\zeta_B(s) := \sum_{I \subset B} \mathbf{N}(I)^{-s} = \prod_{\mathfrak{P}}(1 - \mathbf{N}(\mathfrak{P})^{-s})^{-1}$ on $\Re(s) > 1$, where the sum is over all non-zero ideals in B, and the product is over all non-zero prime ideals in B.

These zeta functions both have a simple pole at $s = 1$, and are rational functions in q^{-s}.

Then the L-function $L(s, \chi_m)$ satisfies

$$\zeta_B(s) = \zeta_A(s) \cdot L(s, \chi_m).$$

L-functions

We use the symbol $\square$ to denote square polynomials. Let γ be a fixed generator of $\mathbb{F}_q^{\times}$, and ℓ be a positive integer. We are interested in considering the following averaging problems:

1. Summing over all non-square monic polynomials ("discriminants"):

$$\mathcal{L}(s,M,\ell)_{\mathcal{R}} := \sum_{\substack{m\in A^+:\\ \deg m=M}} L(s,\chi_m)^{\ell}, \qquad \text{if } M \text{ is an odd integer;}$$

$$\mathcal{L}(s,M,\ell)_{\mathcal{S}} := \sum_{\substack{m\in A^+:\\ \deg m=M,\, m\neq\square}} L(s,\chi_m)^{\ell}, \qquad \text{if } M \text{ is an even integer;}$$

$$\mathcal{L}(s,M,\ell)_{\mathcal{I}} := \sum_{\substack{m\in A^+:\\ \deg m=M,\, m\neq\square}} L(s,\chi_{\gamma\cdot m})^{\ell}, \qquad \text{if } M \text{ is an even integer.}$$

We use the symbol $\square$ to denote square polynomials. Let γ be a fixed generator of $\mathbb{F}_q^\times$, and ℓ be a positive integer. We are interested in considering the following averaging problems:

1. Summing over all non-square monic polynomials ("discriminants"):

$$\mathcal{L}(s,M,\ell)_{\mathcal{R}} := \sum_{\substack{m\in A^+:\\ \deg m=M}} L(s,\chi_m)^\ell, \qquad \text{if } M \text{ is an odd integer;}$$

$$\mathcal{L}(s,M,\ell)_{\mathcal{S}} := \sum_{\substack{m\in A^+:\\ \deg m=M,\, m\neq\square}} L(s,\chi_m)^\ell, \qquad \text{if } M \text{ is an even integer;}$$

$$\mathcal{L}(s,M,\ell)_{\mathcal{I}} := \sum_{\substack{m\in A^+:\\ \deg m=M,\, m\neq\square}} L(s,\chi_{\gamma\cdot m})^\ell, \qquad \text{if } M \text{ is an even integer.}$$

L-functions

2. Summing over all square-free monic polynomials ("fundamental discriminants"):

$$\mathcal{L}^*(s, M, \ell)_{\mathcal{R}} := \sum_{\substack{m \in A^+: \\ \deg m = M}}^{*} L(s, \chi_m)^{\ell}, \quad \text{if } M \text{ is an odd integer;}$$
$$\mathcal{L}^*(s, M, \ell)_{\mathcal{S}} := \sum_{\substack{m \in A^+: \\ \deg m = M}}^{*} L(s, \chi_m)^{\ell}, \quad \text{if } M \text{ is an even integer;}$$
$$\mathcal{L}^*(s, M, \ell)_{\mathcal{I}} := \sum_{\substack{m \in A^+: \\ \deg m = M}}^{*} L(s, \chi_{\gamma \cdot m})^{\ell}, \quad \text{if } M \text{ is an even integer.}$$

Here $*$ means that the sum in question runs over all square-free monic polynomials.

L-functions

Remark

For any odd integer M,

$$\mathcal{L}(s, M, \ell)_{\mathcal{R}} = \sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_{\gamma \cdot m})^{\ell}, \text{ and } \mathcal{L}^*(s, M, \ell)_{\mathcal{R}} = \sideset{}{^*}\sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_{\gamma \cdot m})^{\ell}.$$

Except for the above two families, J. C. Andrade, S. Bae and H. Jung also study averaging values of quadratic L-functions which runs over all prime polynomials.

L-functions

Remark

For any odd integer M,

$$\mathcal{L}(s,M,\ell)_{\mathcal{R}} = \sum_{\substack{m\in A^+:\\ \deg m=M}} L(s,\chi_{\gamma\cdot m})^{\ell}, \text{ and } \mathcal{L}^*(s,M,\ell)_{\mathcal{R}} = \sideset{}{^*}\sum_{\substack{m\in A^+:\\ \deg m=M}} L(s,\chi_{\gamma\cdot m})^{\ell}.$$

Except for the above two families, J. C. Andrade, S. Bae and H. Jung also study averaging values of quadratic L-functions which runs over all prime polynomials.

Questions

Let $\star$ be $\mathcal{R}$, $\mathcal{S}$, or $\mathcal{I}$ and ℓ be a positive integer.

1. **Question:** What are asymptotic formulas of $\mathcal{L}(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?
2. **Question:** What are asymptotic formulas of $\mathcal{L}^*(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?

Questions

Let $\star$ be $\mathcal{R}$, $\mathcal{S}$, or $\mathcal{I}$ and ℓ be a positive integer.

1. **Question:** What are asymptotic formulas of $\mathcal{L}(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?
2. **Question:** What are asymptotic formulas of $\mathcal{L}^*(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?

Questions

Let $\star$ be $\mathcal{R}$, $\mathcal{S}$, or $\mathcal{I}$ and ℓ be a positive integer.

1. **Question:** What are asymptotic formulas of $\mathcal{L}(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?
2. **Question:** What are asymptotic formulas of $\mathcal{L}^*(s, M, \ell)_\star$ for $\Re(s) \geq 1$, as $M \to \infty$?

The non-square case:

In 1992, J. Hoffstein and M. Rosen proved that

Theorem (Journal für die reine und angewandte math, 1992)

For any odd integer M, For $s \neq 1/2$, we have

$$q^{-M}\mathcal{L}(s, M, 1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - (1 - q^{-1})(q^{1-2s})^{\frac{M+1}{2}}\zeta_A(2s).$$

If $\Re(s) > 1/2$, then the above asymptotic averaging value is

$$q^{-M}\mathcal{L}(s, M, 1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

Moreover, they also proved that, for $\star = \mathcal{S}$ or $\mathcal{I}$,

$$q^{-M}\mathcal{L}(s, M, 1)_{\star} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

The non-square case:

In 1992, J. Hoffstein and M. Rosen proved that

Theorem (Journal für die reine und angewandte math, 1992)

For any odd integer M, For $s \neq 1/2$, we have

$$q^{-M}\mathcal{L}(s,M,1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - (1-q^{-1})(q^{1-2s})^{\frac{M+1}{2}}\zeta_A(2s).$$

If $\Re(s) > 1/2$, then the above asymptotic averaging value is

$$q^{-M}\mathcal{L}(s,M,1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

Moreover, they also proved that, for $\star = \mathcal{S}$ or $\mathcal{I}$,

$$q^{-M}\mathcal{L}(s,M,1)_{\star} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

The non-square case:

In 1992, J. Hoffstein and M. Rosen proved that

Theorem (Journal für die reine und angewandte math, 1992)

For any odd integer M, For $s \neq 1/2$, we have

$$q^{-M}\mathcal{L}(s,M,1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - (1-q^{-1})(q^{1-2s})^{\frac{M+1}{2}}\zeta_A(2s).$$

If $\Re(s) > 1/2$, then the above asymptotic averaging value is

$$q^{-M}\mathcal{L}(s,M,1)_{\mathcal{R}} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

Moreover, they also proved that, for $\star = \mathcal{S}$ or $\mathcal{I}$,

$$q^{-M}\mathcal{L}(s,M,1)_{\star} = \frac{\zeta_A(2s)}{\zeta_A(2s+1)}, \text{ as } M \to \infty.$$

The non-square case:

Theorem (C.)

Let ℓ, M be positive integers and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$, then we have, for $\Re(s) \geq 1$,

$$\mathcal{L}(s, M, \ell)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) \cdot q^M + O\left(q^{(1/2+\delta)M}\right),$$

for any $\delta > 0$, as $M \to \infty$. Here $c_\ell(s) :=$

$$\prod_P \Bigg\{ (1 - q^{-\deg P}) \left(\frac{(1 + q^{-s \deg P})^\ell + (1 - q^{-s \deg P})^\ell}{2} \right) + q^{-\deg P} (1 - q^{-2s \deg P})^\ell (1 - q^{-2s \deg P})^{\frac{\ell(\ell-1)}{2}} \Bigg\}$$

is absolutely convergent on $\Re(s) > 1/4$.

The non-square case:

- When $\ell = 1$,

$$c_1(s) = \zeta_A(2s+1)^{-1}$$

  is a rational function in q^{-s}. The above theorem reduces to the averaging L-values of J. Hoffstein and M. Rosen.

- Let D denote an integer congruent to 0 or 1 modulo 4 and non-square. Let $\psi_D(n) := \left(\frac{D}{n}\right)$ denote the Kronecker symbol. The Dirichlet L-function associated with ψ_D is given by

$$L(s, \psi_D) := \sum_{n=1}^{\infty} \psi_D(n)/n^s, \text{ on } \Re(s) > 1.$$

The non-square case:

- When $\ell = 1$,

$$c_1(s) = \zeta_A(2s+1)^{-1}$$

  is a rational function in q^{-s}. The above theorem reduces to the averaging L-values of J. Hoffstein and M. Rosen.

- Let D denote an integer congruent to 0 or 1 modulo 4 and non-square. Let $\psi_D(n) := \left(\frac{D}{n}\right)$ denote the Kronecker symbol. The Dirichlet L-function associated with ψ_D is given by

$$L(s, \psi_D) := \sum_{n=1}^{\infty} \psi_D(n)/n^s, \text{ on } \Re(s) > 1.$$

The non-square case:

Concerning higher moments, M. B. Barban established the following asymptotic formula, for any fixed positive integer ℓ,

$$\sum_{-N\le D\le -1} L^{\ell}(1,\psi_D) = r_\ell N + O(N\exp(-c\sqrt{N})), \text{ as } N\to\infty,$$

where $c>0$ is a constant independent of ℓ and

$$r_\ell = \sum_{\substack{n=1 \\ n\equiv 1 \mod 2}}^{\infty} \frac{\varphi(n)d_\ell(n^2)}{n^3}.$$

Here $d_\ell(n)$ is the number of ways of expressing n as the product ℓ positive integers, expressions in which only the order of the factors being different is regarded as distinct, and $\varphi(n)$ is the Euler totient function.

The non-square case:

In the function field case, the constant of our main term is

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) = \sum_{n\in A^+} \frac{d_\ell(n^2)\cdot\varphi(n)}{|n|^{2s+1}}.$$

Here $|n| := q^{\deg n}$ for any $n \in A - \{0\}$.

In particular, if $s = 1$, then

$$\zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(1) = \sum_{n\in A^+} \frac{d_\ell(n^2)\cdot\varphi(n)}{|n|^{3}}.$$

Here $d_\ell(n)$ is the number of ways of expressing n as the product ℓ monic polynomials, expressions in which only the order of the factors being different is regarded as distinct, and $\varphi(n)$ is the Euler totient function for A.

The non-square case:

In the function field case, the constant of our main term is

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) = \sum_{n \in A^+} \frac{d_\ell(n^2) \cdot \varphi(n)}{|n|^{2s+1}}.$$

Here $|n| := q^{\deg n}$ for any $n \in A - \{0\}$.
In particular, if $s = 1$, then

$$\zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(1) = \sum_{n \in A^+} \frac{d_\ell(n^2) \cdot \varphi(n)}{|n|^3}.$$

Here $d_\ell(n)$ is the number of ways of expressing n as the product ℓ monic polynomials, expressions in which only the order of the factors being different is regarded as distinct, and $\varphi(n)$ is the Euler totient function for A.

Square-free cases: When $\ell = 1$ and $q \equiv 1 \mod 4$, Hoffstein-Rosen uses the fact that the Fourier coefficients of Eisenstein series for the metaplectic group involve the values $L(s, \chi_m)$.

Let

$$P(s) := \prod_{P} \left(1 - |P|^{-2} - |P|^{-(2s+1)} + |P|^{-(2s+2)}\right), \text{ on } \Re(s) \geq 1/2.$$

Then

Square-free cases: When $\ell = 1$ and $q \equiv 1 \mod 4$, Hoffstein-Rosen uses the fact that the Fourier coefficients of Eisenstein series for the metaplectic group involve the values $L(s, \chi_m)$.

Let

$$P(s) := \prod_{P} \left(1 - |P|^{-2} - |P|^{-(2s+1)} + |P|^{-(2s+2)}\right), \text{ on } \Re(s) \geq 1/2.$$

Then

The square-free case:

Theorem (Journal für die reine und angewandte math, 1992)

Choose any $\epsilon > 0$. If $\Re(s) \geq 1$, then

(1). If M is odd, then

$$(q-1)^{-1}(q^M - q^{M-1})^{-1}\sum L(s,\chi_m) = \zeta_A(2)\zeta_A(2s)P(s) + O(q^{-(M/2)(1-\epsilon)}),$$

where the sum is over all square-free m such that $\deg(m) = M$.

(2). If M is even, then

$$2(q-1)^{-1}(q^M - q^{M-1})^{-1}\sum L(s,\chi_m) = \zeta_A(2)\zeta_A(2s)P(s) + O(q^{-(M/2)(1-\epsilon)}),$$

where the sum is over all square-free m such that $\deg(m) = M$ and $\mathrm{sgn}(m) \in (\mathbb{F}_q^\times)^2$ or over all square-free m such that $\deg(m) = M$ and $\mathrm{sgn}(m) \notin (\mathbb{F}_q^\times)^2$.

The square-free case:

1. When $\ell = 1$ and $q \equiv 1 \mod 4$, J. C. Andrade proved that (Int. J. Number Theory 2012):

$$(q^M - q^{M-1})^{-1}\mathcal{L}^*(1, M, \ell)_{\mathcal{R}} = P(1) \cdot \zeta_A^2(2), \text{ as } M \to \infty.$$

Here for $M > 1$, $q^M \zeta_A(2) =$

$$q^M - q^{M-1} = \#\{m \in A^+ : \deg m = M \text{ and } m \text{ is square-free}\}.$$

2. Let $\star$ be $\mathcal{S}$ or $\mathcal{I}$. H. Jung uses the same method to prove that (the Korea J. of Math., 2013 and Int. J. Number Theory, 2014)

$$(q^M - q^{M-1})^{-1}\mathcal{L}^*(1, M, \ell)_{\star} = P(1) \cdot \zeta_A^2(2), \text{ as } M \to \infty.$$

Meanwhile, he also remarks that the above asymptotic formulas holds for any odd prime power q.

The square-free case:

1. When $\ell = 1$ and $q \equiv 1 \mod 4$, J. C. Andrade proved that (Int. J. Number Theory 2012):

$$(q^M - q^{M-1})^{-1}\mathcal{L}^*(1, M, \ell)_{\mathcal{R}} = P(1) \cdot \zeta_A^2(2), \text{ as } M \to \infty.$$

Here for $M > 1$, $q^M \zeta_A(2) =$

$$q^M - q^{M-1} = \#\{m \in A^+ : \deg m = M \text{ and } m \text{ is square-free}\}.$$

2. Let $\star$ be $\mathcal{S}$ or $\mathcal{I}$. H. Jung uses the same method to prove that (the Korea J. of Math., 2013 and Int. J. Number Theory, 2014)

$$(q^M - q^{M-1})^{-1}\mathcal{L}^*(1, M, \ell)_{\star} = P(1) \cdot \zeta_A^2(2), \text{ as } M \to \infty.$$

Meanwhile, he also remarks that the above asymptotic formulas holds for any odd prime power q.

The square-free case:

Theorem (C.)

Let ℓ, M be positive integers. Suppose that $\star$ is either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Then, for $\Re(s) \geq 1$,

$$\mathcal{L}^*(s, M, \ell)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(s) \cdot (q^M \cdot \zeta_A(2)^{-1}) + O\left(q^{(1/2+\delta)M}\right)$$

for any $\delta > 0$, as $M \to \infty$. Here $c_\ell^*(s) :=$

$$\prod_P \Big\{ \frac{\left(1 - q^{-2s \deg P}\right)^{\frac{\ell(\ell+1)}{2}}}{1 + q^{-\deg P}} \Big(\Big(\frac{(1 + q^{-s \deg P})^{-\ell} + (1 - q^{-s \deg P})^{-\ell}}{2} \Big)$$
$$+ q^{-\deg P} \Big) \Big\},$$

which is absolutely convergent in $\Re(s) > 1/4$

The square-free case:

- When $\ell = 1$, we have

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_1^*(s) = \zeta_A(2) \cdot \zeta_A(2s) \cdot \prod_P (1 - |P|^{-2} - |P|^{-2s-1} + |P|^{-2s-2}).$$

Remark

Assume that $q \equiv 1 \mod 4$.

- For $\ell = 1$ and $\star = \mathcal{I}, \mathcal{R}$ or $\mathcal{S}$, J. Hoffstein and M. Rosen got asymptotic formulas of $\mathcal{L}^*(s, M, \ell)_\star$ for $\Re(s) \geq 1/2$.
- $\ell = 1$, J. C. Andrade and J. P. Keating gave more precisely asymptotic formula of $\mathcal{L}^*(1/2, M, \ell)_\mathcal{R}$.
- A. Florea got asymptotic formulas for $\mathcal{L}^*(1/2, M, \ell)_\mathcal{R}$ for $\ell = 2$ and 3.

The square-free case:

- When $\ell = 1$, we have

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_1^*(s) = \zeta_A(2) \cdot \zeta_A(2s) \cdot \prod_P (1 - |P|^{-2} - |P|^{-2s-1} + |P|^{-2s-2}).$$

Remark

Assume that $q \equiv 1 \mod 4$.

- For $\ell = 1$ and $\star = \mathcal{I}, \mathcal{R}$ or $\mathcal{S}$, J. Hoffstein and M. Rosen got asymptotic formulas of $\mathcal{L}^*(s, M, \ell)_\star$ for $\Re(s) \geq 1/2$.
- $\ell = 1$, J. C. Andrade and J. P. Keating gave more precisely asymptotic formula of $\mathcal{L}^*(1/2, M, \ell)_{\mathcal{R}}$.
- A. Florea got asymptotic formulas for $\mathcal{L}^*(1/2, M, \ell)_{\mathcal{R}}$ for $\ell = 2$ and 3.

Applications:

Let $K := k(\sqrt{m})$ be a quadratic field over k, where m is non-square with $\deg m \geq 1$. Write $m = m_0 m_1^2$, where m_0 is square-free. Define B_m to be the ring $A + Am_1\sqrt{m_0} \subset K = k(\sqrt{m})$. It is an A-order in K, (i.e. it is a ring, finitely generated as an A-module, and its quotient field is K). The Picard group $\mathrm{Pic}(B_m)$ is the group of invertible fractional ideals of B_m modulo the subgroup of principal fractional ideals. We set the class number $h_m := \#\,\mathrm{Pic}(B_m)$.

If ∞ doesn't split in K/k, then $B_m^\times = \mathbb{F}_q^\times$, and if ∞ splits in K/k, then $B_m^\times = \mathbb{F}_q^\times \times < \epsilon_m >$, where $< \epsilon_m >$ is infinite cyclic. In this case, we set R_m equal to the absolute value of $\log q^{\mathrm{ord}_\infty(\epsilon_m)}$.

Applications:

Let $K := k(\sqrt{m})$ be a quadratic field over k, where m is non-square with $\deg m \geq 1$. Write $m = m_0 m_1^2$, where m_0 is square-free. Define B_m to be the ring $A + Am_1\sqrt{m_0} \subset K = k(\sqrt{m})$. It is an A-order in K, (i.e. it is a ring, finitely generated as an A-module, and its quotient field is K). The Picard group $\mathrm{Pic}(B_m)$ is the group of invertible fractional ideals of B_m modulo the subgroup of principal fractional ideals. We set the class number $h_m := \# \mathrm{Pic}(B_m)$.

If ∞ doesn't split in K/k, then $B_m^\times = \mathbb{F}_q^\times$, and if ∞ splits in K/k, then $B_m^\times = \mathbb{F}_q^\times \times < \epsilon_m >$, where $< \epsilon_m >$ is infinite cyclic. In this case, we set R_m equal to the absolute value of $\log q^{\mathrm{ord}_\infty(\epsilon_m)}$.

Applications:

Theorem (Artin)

Let $m \in A$ be a non-square polynomial of degree $M \geq 1$.

(1). $L(1, \chi_m) = q^{\frac{1-M}{2}} \cdot h_m$, if M is odd.

(2). $L(1, \chi_m) = \frac{q+1}{2} \cdot q^{-M/2} \cdot h_m$, if M is even and $\mathrm{sgn}_2(m) = -1$.

(3). $L(1, \chi_m) = (q-1) \cdot q^{-M/2} \cdot h_m \cdot R_m$, if M is even and $\mathrm{sgn}_2(m) = 1$. Here R_m is the regulator of the ring B_m.

Applications:

Corollary

Let ℓ be a positive integer. Then, for any $\delta > 0$,

(1). If $\deg m = M$ is an odd integer, then

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} h_m^{\ell} = q^{\left(1+\frac{\ell}{2}\right)M} \cdot q^{-\ell/2} \cdot \zeta_A(2)^{\frac{\ell(\ell+1)}{2}-1} \cdot c_\ell^*(1) + O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

(2). If $\deg m = M$ is an even integer, and γ is a generator of $\mathbb{F}_q^\times$, then

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} h_{\gamma m}^{\ell} = q^{\left(1+\frac{\ell}{2}\right)M} \cdot \left(\frac{2}{q+1}\right)^{\ell/2} \cdot \zeta_A(2)^{\frac{\ell(\ell+1)}{2}-1} \cdot c_\ell^*(1) + O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

(3). If $\deg m = M$ is an even integer, then

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} (h_m \cdot R_m)^{\ell} = q^{\left(1+\frac{\ell}{2}\right)M}\left(\frac{1}{q-1}\right)^{\ell/2}\zeta_A(2)^{\frac{\ell(\ell+1)}{2}-1} \cdot c_\ell^*(1) + O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

Applications:

Let F be a quadratic extension over $\mathbb{Q}$. Let $\Delta_{F/\mathbb{Q}}$, h_F, and R_F be the discriminant of $F/\mathbb{Q}$, the class number, and the regulator of F, respectively.

In 2008, T. Taniguchi conjectured that (Ann. Inst. Fourier, 2008):

$$\lim_{X\to\infty} \frac{1}{X^2} \cdot \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<\left|\Delta_{F/\mathbb{Q}}\right|\leq X}} h_F^2 \cdot R_F^2 = \frac{\zeta(2)^2}{2^4} \cdot \prod_p \left(1 - \frac{3}{p^3} + \frac{2}{p^4} + \frac{1}{p^5} - \frac{1}{p^6}\right).$$

Applications:

Let F be a quadratic extension over $\mathbb{Q}$. Let $\Delta_{F/\mathbb{Q}}$, h_F, and R_F be the discriminant of $F/\mathbb{Q}$, the class number, and the regulator of F, respectively.

In 2008, T. Taniguchi conjectured that (Ann. Inst. Fourier, 2008):

$$\lim_{X\to\infty} \frac{1}{X^2} \cdot \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<\left|\Delta_{F/\mathbb{Q}}\right|\leq X}} h_F^2 \cdot R_F^2 = \frac{\zeta(2)^2}{2^4} \cdot \prod_p \left(1 - \frac{3}{p^3} + \frac{2}{p^4} + \frac{1}{p^5} - \frac{1}{p^6}\right).$$

When $\ell = 2$, simplifying our cases (1), (2), and (3) in the above , we have proved

$$\begin{aligned}
&(1) + (2) + (3) \\
=&q^{\left(1+\frac{\ell}{2}\right)M} \cdot \left(\frac{1}{q-1} + \frac{2}{q+1} + \frac{1}{q} \right) \cdot \zeta_A(2)^{-1} \cdot \zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_2^*(1) \\
=&\left(\frac{1}{q-1} + \frac{2}{q+1} + \frac{1}{q} \right) \cdot \zeta_A(2)^2 \\
&\cdot \prod_P \left(1 - \frac{3}{|P|^3} + \frac{2}{|P|^4} + \frac{1}{|P|^5} - \frac{1}{|P|^6} \right) \cdot q^{2M},
\end{aligned}$$

which is to be compared with T. Taniguchi's conjecture.

Applications:

- A distribution function is a non-decreasing function $f : \mathbb{R} \to [0,1]$ which is right continuous and satisfies $f(-\infty) = 0$ and $f(\infty) = 1$.
- The characteristic function of f is the Fourier transform of the Stieltjes measure $df(z)$, viz.

$$\varphi_f(s) := \int_{-\infty}^{\infty} e^{isz}\, df(z), \text{ where } s \in \mathbb{R}.$$

Applications:

- A distribution function is a non-decreasing function $f : \mathbb{R} \to [0,1]$ which is right continuous and satisfies $f(-\infty) = 0$ and $f(\infty) = 1$.
- The characteristic function of f is the Fourier transform of the Stieltjes measure $df(z)$, viz.

$$\varphi_f(s) := \int_{-\infty}^{\infty} e^{isz} df(z), \text{ where } s \in \mathbb{R}.$$

Applications:

Corollary

For real $s_0 \geq 1$, as $M \to \infty$, the quantity $f_M^*(x, s_0) :=$

$$\frac{\#\{m \in A^+, \deg m = M \text{ and } m \text{ is square-free}: \ L(s_0, \chi_m) \leq x\}}{q^M - q^{M-1}}, \ x \in \mathbb{R}$$

converges to a distribution function $f^*(x) := f^*(x, s_0)$ at each point of continuity of the latter, and the corresponding characteristic function has the form

$$\phi_{f,s_0}^*(x) = 1 + \sum_{\ell \geq 1} \frac{r_\ell^*(s_0)}{\ell!}(ix)^\ell, \ x \in \mathbb{R}.$$

Here $r_\ell^*(s) := \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_1^*(s)$. Recall that

$$\mathcal{L}^*(s, M, \ell)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(s) \cdot (q^M \cdot \zeta_A(2)^{-1}) + O\left(q^{(1/2+\delta)M}\right).$$

A type of quadratic Gauss sum:

Let k_∞ be the completion field of k at ∞, O_∞ be the valuation ring of k_∞, and $\pi_\infty = t^{-1}$ be a fixed uniformizer. We fix an additive character ψ_∞ of k_∞ as the following, for $y := \sum_{i=N}^{\infty} a_i \pi_\infty^i \in k_\infty$,

$$\psi_\infty(y) := \exp(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)) \in \mathbb{C}^\times.$$

For a locally constant function with compact support $f : k_\infty \to \mathbb{C}$, the Fourier transform $\hat{f}$ of f is defined to be

$$\hat{f}(y) := \int_{k_\infty} f(x)\psi_\infty(xy)dx,$$

where dx is the Haar measure of k_∞ such that $\hat{\hat{f}}(x) = f(-x)$ holds.

A type of quadratic Gauss sum:

Let k_∞ be the completion field of k at ∞, O_∞ be the valuation ring of k_∞, and $\pi_\infty = t^{-1}$ be a fixed uniformizer. We fix an additive character ψ_∞ of k_∞ as the following, for $y := \sum_{i=N}^{\infty} a_i \pi_\infty^i \in k_\infty$,

$$\psi_\infty(y) := \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)\right) \in \mathbb{C}^\times.$$

For a locally constant function with compact support $f : k_\infty \to \mathbb{C}$, the Fourier transform $\hat{f}$ of f is defined to be

$$\hat{f}(y) := \int_{k_\infty} f(x)\psi_\infty(xy)dx,$$

where dx is the Haar measure of k_∞ such that $\hat{\hat{f}}(x) = f(-x)$ holds.

A type of quadratic Gauss sum:

Let n be a monic polynomial. For all polynomials $e \in A$, we define an analogue of Gauss sum as follows:

$$G_e(n) := \sum_{a \mod n} \left[\frac{a}{n}\right] \psi_\infty\left(-\frac{ae}{n}\right) \in \mathbb{C},$$

and put

$$\tilde{G}_e(n) := \left(\frac{1+i}{2} + \left[\frac{-1}{n}\right]\frac{1-i}{2}\right) G_e(n).$$

Here $i := \sqrt{-1}$.

A type of quadratic Gauss sum:

Lemma

(1). Suppose m and n are co-prime monic polynomials. Then

$$\tilde{G}_e(mn) = \tilde{G}_e(m)\tilde{G}_e(n).$$

(2). Suppose that $d \in A$, and α is the largest power of irreducible polynomial P dividing e (If $e = 0$ then set $\alpha = \infty$). Then for $\beta \geq 1$

$$\tilde{G}_e(P^\beta) := \begin{cases} 0, & \text{if } \beta \leq \alpha \text{ is odd;} \\ \varphi(P^\beta), & \text{if } \beta \leq \alpha \text{ is even;} \\ -q^{\alpha \deg P}, & \text{if } \beta = \alpha + 1 \text{ is even;} \\ (\gamma_{p,q})^{\deg P} \cdot \left[\frac{eP^{-\alpha}}{P}\right] \cdot q^{(\alpha+1/2)\deg P}, & \text{if } \beta = \alpha + 1 \text{ is odd;} \\ 0, & \text{if } \beta \geq \alpha + 2. \end{cases}$$

A type of quadratic Gauss sum:

Here

$$\gamma_{p,q} := -\left(-\sqrt{\left(\frac{-1}{p}\right)}\right)^{[\mathbb{F}_q:\mathbb{F}_p]} \cdot \left(\frac{1+i}{2} + (-1)^{\frac{q-1}{2}}\frac{1-i}{2}\right) \in \{\pm 1\}.$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p, and $[\mathbb{F}_q : \mathbb{F}_p]$ is the dimension of $\mathbb{F}_q$ over $\mathbb{F}_p$.

Rearrange our sums

Let ℓ be a positive integer and M be an odd integer, then we have

$$\mathcal{L}(s,M,\ell)_{\mathcal{R}} = \sum_{\substack{m\in A^{+}:\\ \deg m=M}} \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^{+}:\\ \deg n=N}} d_{\ell}(n)\chi_m(n) \right) q^{-sN}$$

$$= \sum_{\substack{N=0\\ 2|N}}^{\ell(M-1)} q^{M-N} \sum_{\substack{n\in A^{+}:\\ \deg n=N}} d_{\ell}(n)$$

$$\left(\sum_{\substack{e\in A:\\ \deg e\leq N-M-2}} \tilde{G}_e(n) - \sum_{\substack{e\in A^{+}:\\ \deg e=N-M-1}} \tilde{G}_e(n) \right) q^{-sN}.$$

The main term

On the basis of the above computation, we divide the following sum

$$\sum_{\substack{e\in A:\\ \deg e\leq N-M-2}}\ \sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\tilde{G}_e(n) - \sum_{\substack{e\in A^+:\\ \deg e=N-M-1}}\ \sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\tilde{G}_e(n)$$

into three parts which are $e=0$, $e=\square$ and $e\neq\square$. The main contribution of our asymptotic formula comes form the case $e=0$.

Thanks

Thank you for your attention!!