Unlikely intersection for two-parameter families of polynomials

joint work with D. Ghioca and T. Tucker

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Taipie-Xi'an Number Theory Workshop
TIMS, October 14, 2015

Outline

Orbits, (pre)periodic points and wandering points

Let

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1 \quad \text{of degree } d \geq 2 \text{ over } \mathbb{C}.$$

Dynamical system associated to φ is roughly the classification of points in $\mathbb{P}^1$ under the action of φ via iterations $\{\varphi^n \mid n \geq 0\}$
Let $\alpha \in \mathbb{P}^1$. The (forward) orbit of α

$$\mathcal{O}_\varphi(\alpha) := \{\alpha, \varphi(\alpha), \varphi^2(\alpha), \ldots\}.$$

α is called *preperiodic* for φ if $\sharp\, \mathcal{O}_\varphi(\alpha) < \infty$;
wandering for φ if $\sharp\, \mathcal{O}_\varphi(\alpha) = \infty$.

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Comparison: dynamical systems v.s. algebraic group G

$$\begin{aligned}
\text{action by } \{\varphi^n\} &\longleftrightarrow \text{action by } \mathrm{End}(G)\\
\text{preperiodic points} &\longleftrightarrow \text{torsion points}\\
\text{wandering points} &\longleftrightarrow \text{point of infinite order}\\
\text{orbit } \mathcal{O}_\varphi(\alpha) &\longleftrightarrow \text{Cyclic subgroup of } G(K).\\
\text{families of dynamical systems} &\longleftrightarrow \text{families of algebraic groups}\\
\text{parameter space} &\longleftrightarrow \text{moduli curves (spaces)}
\end{aligned}$$

Example

$f(x) = x^d$ ($d \geq 2$). Then, the set of nonzero f-preperiodic points are all the the roots of unity. The fields generated by the roots of unity (the cyclotomic fields) play the key roles in many arithmetic theories.

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A special case

Theorem (Y. Ihara, S. Lang, J.-P. Serre, J. Tate,)

Let $T \subset \mathbb{C}^2$ be an algebraic curve. If there are infinitely many points $(x, y) \in T$ such that both x and y are roots of unity, then the equation of T is of the form $X^m Y^n = \zeta$, where $m, n \in \mathbb{Z}$ and $\zeta \in \mu_\infty$ is a root of unity.

The theorem can reformulated as follows.

Theorem

Let T be an algebraic curve and let F_1, F_2 be non-zero rational functions in $\mathbb{C}(T)$ such that there are infinitely many $P \in T(\mathbb{C})$ such that both $F_1(P)$ and $F_2(P)$ are roots of unity. Then, F_1 and F_2 are multiplicative dependent, i.e. there exist $m, n \in \mathbb{Z}$ (not both equal to zero) such that $F_1^m F_2^n = 1$.

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There are higher dimensional analogues of this theorem (i.e. T is an irreducible subvariety of torus $\mathbb{G}_m^n$) with "infinitely many" replaced by "Zariski dense".

Another generalization is the following theorem proved by E. Bombieri, D. Masser and U. Zannier.

Theorem (Bombieri-Masser-Zannier)

Let T be an absolutely irreducible curve defined over $\bar{\mathbb{Q}}$ and let $x_1, \ldots, x_n$ be nonzero rational functions in $\bar{\mathbb{Q}}(T)$, multiplicaitve independent modulo constants. Then the points $P \in T(\bar{\mathbb{Q}})$ for which $x_1(P), \ldots, x_n(P)$ are multiplicative dependent, form a set of bounded height.

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Legendre family of elliptic curves

Consider the one-parameter family of elliptic curves E_t given by the following Weierstrass equation

$$E_t : y^2 = x(x-1)(x-t), \quad \text{(the Legendre family)}$$

and the two points

$$P = P_t = (2, \sqrt{4-2t}), \qquad Q = Q_t = (3, \sqrt{18-6t}).$$

Question

Are there infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points of E_λ?

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Question

Are there infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points of E_λ?

- P and Q are viewed as points in $E_t(\overline{\mathbb{C}(t)})$.
- It's not hard to see that the set of parameters $\mathcal{T}(P) := \{\lambda \in \mathbb{C} \mid P_\lambda$ is a torsion of $E_\lambda\}$ (and $\mathcal{T}(Q)$ respectively) is an infinite set.
- On the other hand, neither P nor Q is a torsion point on the elliptic surface E_t (over $\mathbb{C}(t)$).
- Also, the two pionts P and Q are independent points of $E_t(\overline{\mathbb{C}(t)})$.

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Masser-Zannier's result

Theorem (Masser-Zannier)

Let E_t, P_t and Q_t be given as above. Then, the intersection $\mathcal{T}(P) \cap \mathcal{T}(Q)$ is a finite set.

More general version of this theorem was also obtained (Masser-Zannier, 2011) and a version of the case of higher dimensional base was established (Habegger, 2011).

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(One parameter) family of polynomials

Theorem (Baker-DeMarco)

Let $d \geq 2$ be an integer, and fix $a, b \in \mathbb{C}$. The set of parameters $t \in \mathbb{C}$ such that both a and b are preperiodic *for the polynomial map $P_t(z) = z^d + t$ is infinite if and only if $a^d = b^d$.*

- $a^d = b^d \iff P_t(a) = P_t(b).$

Generalizations ?

- The two points a and b depend on the parameter t algebraically.
- More general family of polynomial maps.
- The case of families of rational maps.

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Family of dynamical systems and specialization

Consider morphisms

$$\mathbf{f} : \mathbb{P}^1 \to \mathbb{P}^1 \text{ of degree } d \geq 2 \text{ over } \mathbb{C}(t)$$

$$t \text{ as an indeterminate over } \mathbb{C}$$

Then, for all but finitely many $t = \lambda \in \mathbb{C}$,

$$\mathbf{f}_\lambda : \mathbb{P}^1 \to \mathbb{P}^1 \quad \text{is of degree } d \text{ over } \mathbb{C}.$$

Let $P, Q \in \mathbb{P}^1(\mathbb{C}(t))$ be given and put

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Characterize P and Q so that $\mathrm{PreP}_{\mathbf{f}}(P, Q)$ is an infinite set.

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Let $P, Q \in \mathbb{P}^1(\mathbb{C}(t))$ be given and put

$$\mathrm{PreP}_{\mathbf{f}}(P) := \{\lambda \in \mathbb{C} \; : \; P_\lambda \text{ is preperiodic for } \mathbf{f}_\lambda\}.$$

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Characterize P and Q so that $\mathrm{PreP}_{\mathbf{f}}(P, Q)$ is an infinite set.

Family of dynamical systems and specialization

Consider morphisms

$$\mathbf{f} : \mathbb{P}^1 \to \mathbb{P}^1 \text{ of degree } d \geq 2 \text{ over } \mathbb{C}(t)$$

$$t \text{ as an indeterminate over } \mathbb{C}$$

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Characterize P and Q so that $\mathrm{PreP}_{\mathbf{f}}(P, Q)$ is an infinite set.

Some sufficient conditions:

Remark

There are two sufficient conditions so that there are infinitely many $t \in \mathbb{C}$ such that both P_t and Q_t are preperiodic for $\mathbf{f}_t$.

- P is preperiodic for $\mathbf{f}$ (i.e. $\mathbf{f}^m(P) = \mathbf{f}^n(P)$ for some non-negative integers m, n).
- $\varphi_1(\mathbf{f}^k(P)) = \varphi_2(\mathbf{f}^\ell(Q))$ for some rational functions $\varphi_i, i = 1, 2$ which commute with a power $\mathbf{f}^m$ of $\mathbf{f}$.

Family of polynomial maps

Theorem (G.-H.-T.)

Let

$$\mathbf{f}(z) := z^d + \sum_{i=0}^{d-2} c_i(t) z^i \in \mathbb{C}[t, z]$$

Let $P, Q \in \mathbb{C}[t]$, *and assume that* $\mathrm{PreP}(P, Q)$ *is infinite then there exists an* $\mathbf{h} \in \mathbb{C}[t, z]$ *and integers* $k > 0, m, n \geq 0$ *such that* $\mathbf{h} \circ \mathbf{f}^k = \mathbf{f}^k \circ \mathbf{h}$ *and* $\mathbf{f}^n(P) = \mathbf{h}(\mathbf{f}^m(Q))$.

Multi-parameter family of polynomials

For algebraically independent variables $t_1, \ldots, t_m$ we define

$$\mathbf{f}_{\mathbf{t}}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m, \; d > m \geq 2$$
$$\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{A}^m(\mathbb{C}).$$

Let $c_1, \ldots, c_\ell \in \mathbb{C}$ be distinct complex numbers.

$$\mathrm{PreP}_{\mathbf{f}}(c_1, \ldots, c_\ell) := \{\boldsymbol{\lambda} \in \mathbb{A}^m(\mathbb{C}) \mid c_i \text{ is preperiodic for } \mathbf{f}_{\boldsymbol{\lambda}}, i = 1, \ldots, \ell\}$$

Question

When is $\mathrm{PreP}_{\mathbf{f}}(c_1, \ldots, c_\ell)$ infinite?

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For $1 \le i \le \ell$, c_i is preperiodic for $\mathbf{f_t}$ if and only if there exist $(m_i, n_i) \in \mathbb{Z}_{\ge 0} \times \mathbb{N}$ such that

$$\mathbf{f}_{\mathbf{t}}^{m_i}(c_i) = \mathbf{f}_{\mathbf{t}}^{n_i}(c_i)$$

which defines a hypersurface S_{m_i,n_i} in $\mathbb{A}^m(\mathbb{C})$.

$$\mathrm{PreP}_{\mathbf{f}}(c_1, \ldots, c_\ell) = \bigcup_{(\mathbb{Z}_{\ge 0} \times \mathbb{N})^\ell} \left(\bigcap_{i=1}^{\ell} S_{m_i,n_i} \right).$$

- If $\ell \le m$, then there are many $\boldsymbol{\lambda} \in \mathbb{A}^m(\mathbb{C})$ such that $c_1, \ldots, c_\ell$ are preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$.

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Example

Consider $\mathbf{f_t}(z) = z^d + t_1 z + t_2$ and let $c \in \mathbb{C}$ be a nonzero complex number and $\zeta \in \mu_{d-1}$ be a non-trivial $(d-1)$-st root of unity.

- There are infinitely many $\mathbf{t} = (t_1, 0) \in \mathbb{C}^2$ such that $0, c, \zeta c$ are preperiodic for $\mathbf{f_t}$. Hence, $\mathrm{PreP}_{\mathbf{f}}(0, c, \zeta c)$ is infinite.
- Our main result (see below) implies that $\mathrm{PreP}_{\mathbf{f}}(0, c, \zeta c)$ is not Zariski dense in $\mathbb{A}^2(\mathbb{C}) = \mathbb{C}^2$.

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Theorem A

Theorem (G.-H.-T.)

Let $c_1, c_2, c_3 \in \mathbb{C}$ be distinct complex numbers, and let $d \geq 3$ be an integer. Then the set of all pairs $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that each c_i is preperiodic for the action of $z \mapsto z^d + \lambda_1 z + \lambda_2$ is not Zariski dense in $\mathbb{C}^2$.

- A key ingredient of the proof is the following multi-parameter family but weaker version of the main theorem.

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Theorem B

Theorem (G.-H.-T.)

Let K be a number field, or a function field of finite transcendence degree over $\bar{\mathbb{Q}}$, let $d > m \geq 2$ be integers, and let

$$\mathbf{f_t}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m.$$

Let $c_1, \ldots, c_{m+1} \in K$ be distinct elements. If $\mathrm{PreP}(c_1, \ldots, c_{m+1})$ is Zariski dense in $\mathbb{A}^m(\bar{K})$ then the following holds: for each $\boldsymbol{\lambda} \in \mathbb{A}^m(\bar{K})$, if m of the points $c_1, \ldots, c_{m+1}$ are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$, then all $(m+1)$ points are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$.

Question

(1). Consider the family $\mathbf{f_t}(z) = z^d + t_1 z + t_2$ and points $0, c, \zeta c$ $(c \neq 0, \zeta \in \mu_{d-1})$. Is the set $\mathrm{PreP}(0, c, \zeta c) \setminus \{(\lambda, 0) \mid \lambda \in \mathbb{C}\}$ finite ?

(2). Characterize $c_1, c_2, c_3 \in \mathbb{C}$ such that $\mathrm{PreP}_{\mathbf{f}}(c_1, c_2, c_3)$ is an infinite set for $\mathbf{f_t}(z) = z^d + t_1 z + t_2$.

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Question

Let $d \geq 3$ be an integer, let $c_1, \ldots, c_d$ be distinct complex numbers, and let $\mathbf{f_t}(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}$ be a family of degree d polynomials in normal form parametrized by $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{A}^{d-1}(\mathbb{C})$. Is it true that the set of parameters $\boldsymbol{\lambda}$ such that each c_i is preperiodic under the action of the polynomial $\mathbf{f}_{\boldsymbol{\lambda}}$ is not Zariski dense in $\mathbb{A}^{d-1}$?

Thank you !