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## Welcome Letter

The first three international conferences: “Nonlinear Waves -- Theory and Applications” took place in Beijing, China in the summers of 2008, 2010 and 2013. These conferences were characterized by their high level of scientific content and attractive venue for collegial interactions. Each of the three conferences had approximately 200 speakers delivering talks on a wide range of topics in nonlinear waves. Detailed information on these conferences can be found at the websites:
http://lsec.cc.ac.cn/~icnwta, http://lsec.cc.ac.cn/~icnwta2/ and http://lsec.cc.ac.cn/~icnwta3/.

Based on the success of these three conferences, we are now organizing the fourth international conference: “Nonlinear waves --- Theory and Applications” in Beijing, China (on the campus of Tsinghua University) from June 25-28, 2016. As before, the goal of this conference is to survey recent advances on a wide range of topics of current interest in nonlinear waves and related phenomena including: integrable and non-integrable

- Organizing Committee
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# Nonlinear Wave Dynamics in PT-symmnmetric Systems

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Joint work with postdoc Sean Nixon

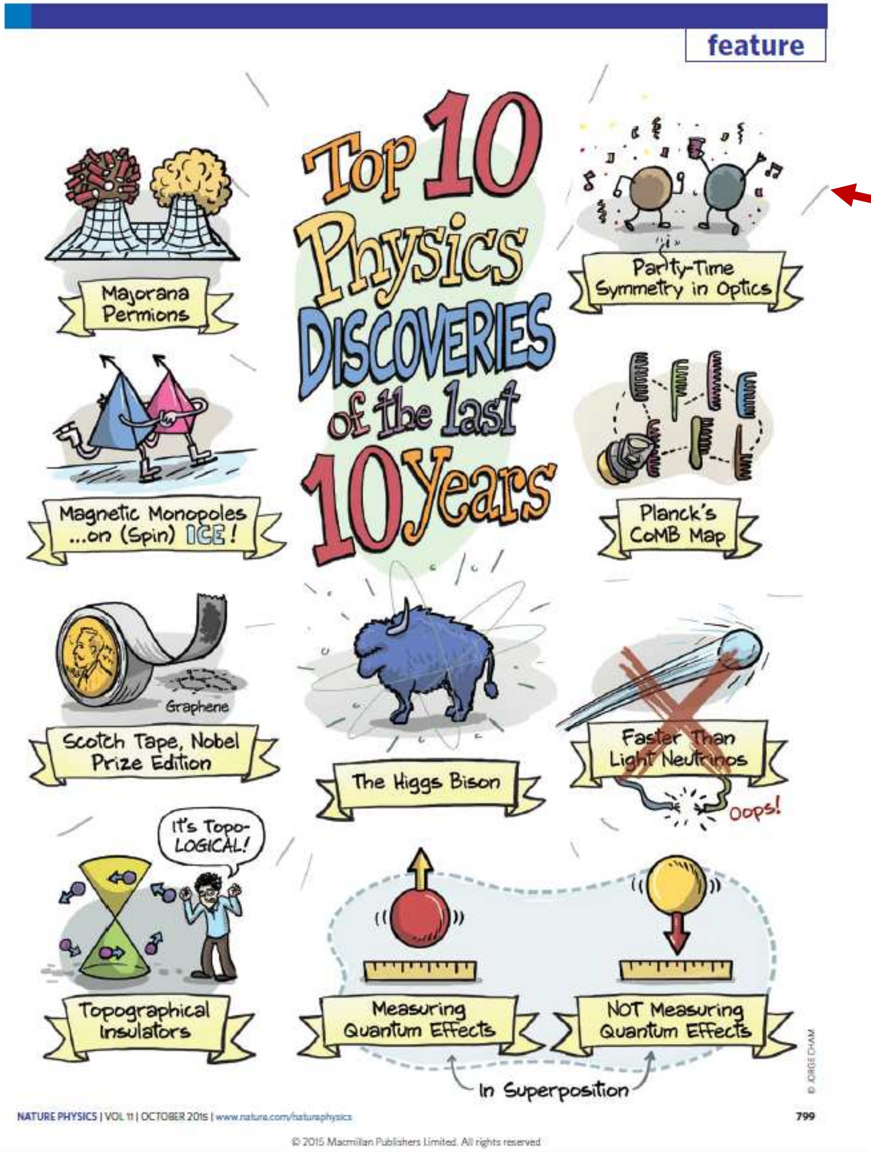
# Talk Outline:

- Introduction
- Phase transition in PT-symmetric media
- Nonlinear dynamics near phase transition
- Symmetry breaking of solitons in PT-symmetric media
- Soliton families in non-PT-symmetric media
- Extension to 2D
- Summary

# 1. Introduction

**PT-symmetry in optics is ranked among**
**top 10 physics discoveries of the last 10 years**

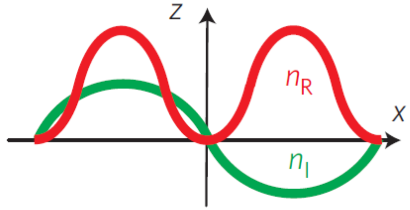
**Nature Physics, October 2015**



# 1. Introduction (cont'd)

**So what is PT symmetry in optics?**

**In optics, PT symmetry mean that the refractive index is even, and the gain-loss profile is odd.**




**Or loosely speaking, PT symmetry in optics means that gain and loss are balanced.**

**The concept of PT symmetry was introduced first by Bender and Boettcher (1998) as a generalization of quantum mechanics.**

**It was later introduced into optics by**

El-Ganainy, Makris, Christodoulides and Musslimani (2007)

# 1. Introduction (cont'd)

**Why is PT symmetry interesting for optical physics?**

**Many reasons:**

(1) **Fundamental new physical phenomena**

(2) **Applications to novel optical devices**

(3) **Enlightening ideas**

(4) **….**

# 1. Introduction (cont'd)

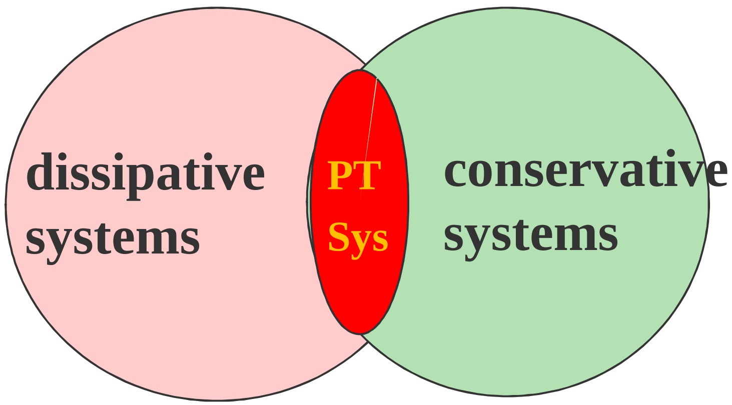
## (1) Fundamental new physical phenomena

PT systems are dissipative due to gain and loss. However, they admit many properties of conservative systems, such a

(a) all real linear spectrum

(b) solitons exist at arbitrary power levels

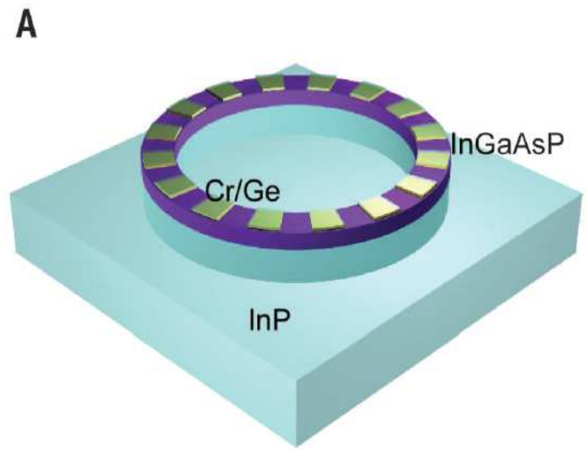
These properties are very surprising.



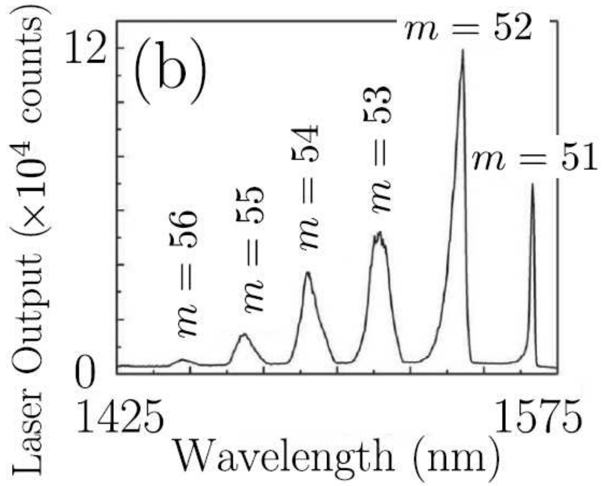
# 1. Introduction (cont’d)

## (2) Applications to novel optical devices

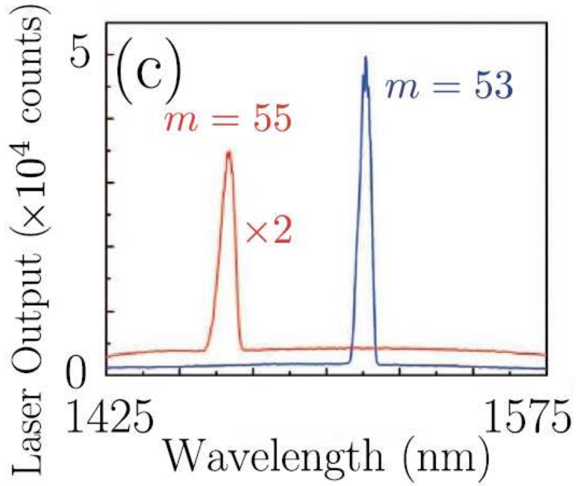
**Single-mode PT lasers (Feng, et al. Science 2014):**




laser device




output without loss

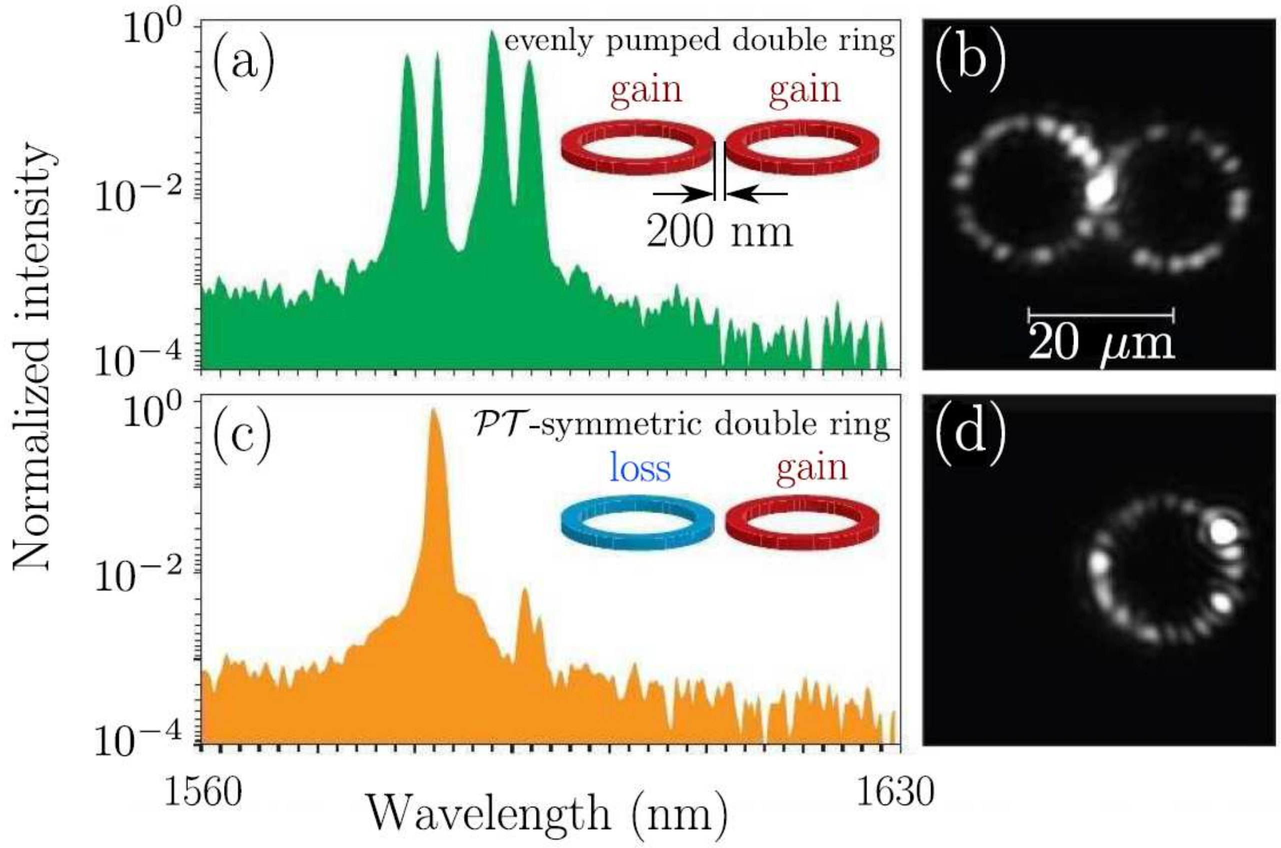



output with PT loss

# 1. Introduction (cont'd)

## (2) Applications to novel optical devices

**Single-mode PT lasers (Hodaei et al. Science 2014):**




**output for both rings with gain**

**output with PT rings**

# 1. Introduction (cont'd)

## (3) Enlightening ideas

**Loss was used to be considered as a detrimental physical effect that should always be suppressed.**

**PT symmetry makes loss useful!**

# 1. Introduction (cont'd)

**Why do we study nonlinear effects in PT systems?**

1. **Because nonlinearity is inherent in many optical systems.**

   **For instance, lasing is well-known to be an intrinsically nonlinear process.**

   **As the laser power amplifies inside the laser cavity, nonlinear effects keep increasing.**

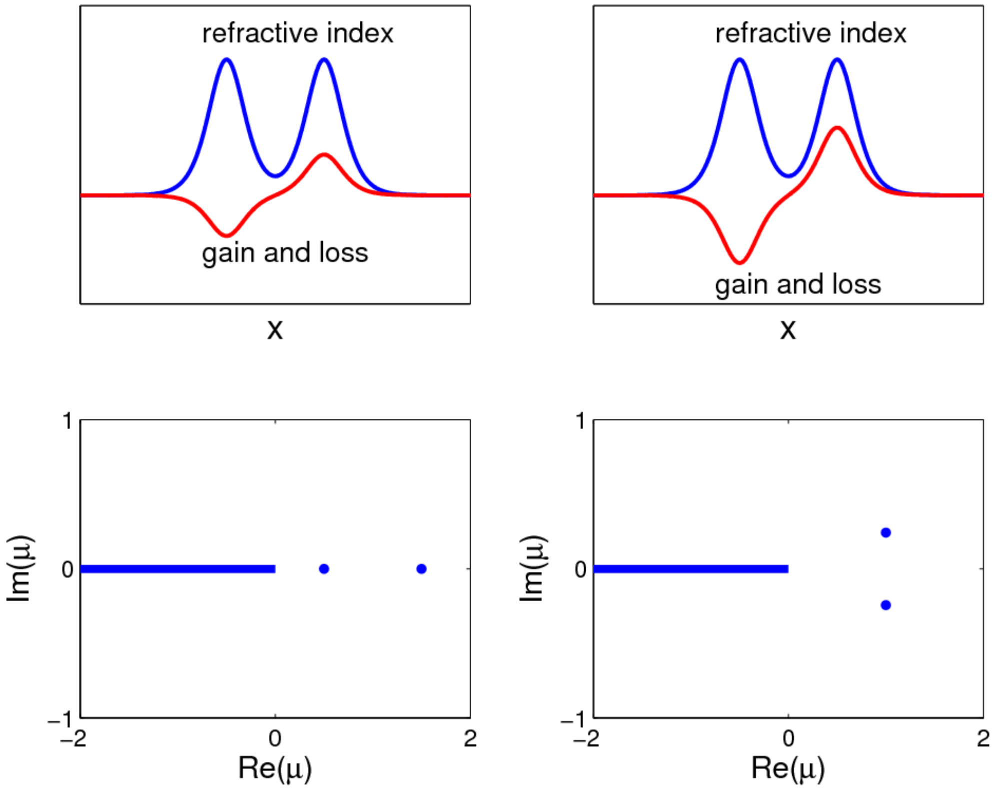
2. **Because nonlinear PT systems exhibit very interesting physical phenomena.**

## 2. Phase transition in PT-symmetric systems

**Currently, all important applications of PT symmetry (such as PT lasers) hinge on the phenomenon of phase transition in PT systems.**

**What is phase transition in PT systems?**

**at lower gain/loss** **at higher gain/loss**

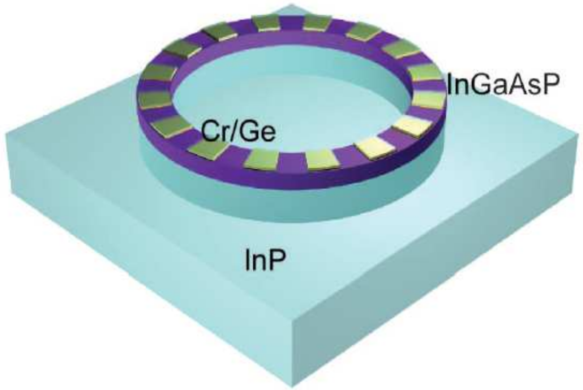



**linear spectrum all-real** **linear spectrum complex**

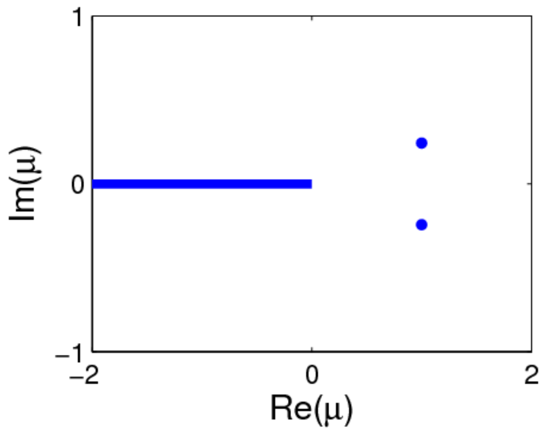
**Phase transition occurs when the linear spectrum changes from all-real to complex.**

# 2. Phase transition in PT-symmetric systems

**In PT lasers,**




**linear spectrum**




**Lasing occurs due to the appearance of complex eigenvalues in the linear spectrum above phase transition.**

**Question: how does lasing light behave when its intensity is high enough to trigger nonlinear effects?**

**This question was not considered in previous experiments.**

## 3. Nonlinear dynamics near phase transition

In the beginning of this talk, we use a simpler mathematical model to study nonlinear light propagation in 1D PT media around phase transition (Nixon and Yang, 2015).

The mathematical model is the well-known NLS equation with a PT-symmetric potential,

$$\mathrm{i}\Psi_z + \Psi_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0,$$

$z$: propagation direction
$x$: transverse direction and
$\sigma = \pm 1$: sign of nonlinearity.

The complex potential $V(x)$ contains gain and loss and is assumed to be $\mathcal{PT}$-symmetric,

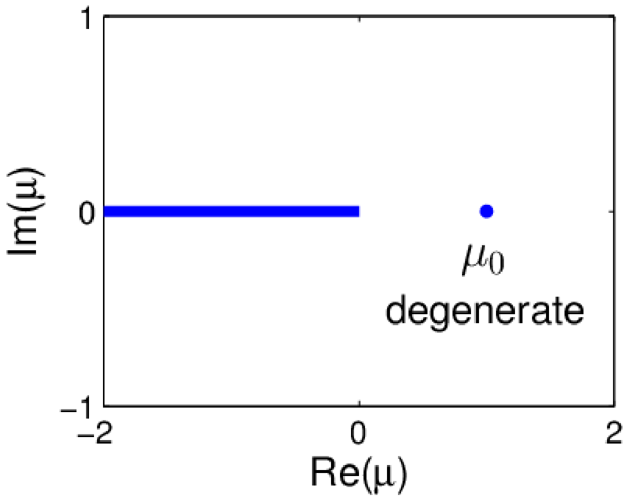
$$V^*(x) = V(-x).$$

# 3. Nonlinear dynamics near phase transition

Why study around phase transition? Because

- phase transition is important in applications
- phase transition allows us to study this nonlinear problem analytically

At phase transition, the linear spectrum is




where the discrete eigenvalue $\mu_0$ is degenerate (with algebraic multiplicity 2 and geometric multiplicity 1).

## 3. Nonlinear dynamics near phase transition

This means that at phase transition, the linear operator

$$L_0 \equiv \partial_{xx} + V_0(x) - \mu_0$$

has an eigenfunction $u_e$ and a generalized eigenfunction $u_g$, where

$$L_0 u_e = 0, \quad L_0 u_g = u_e.$$

Now we consider a perturbed PT potential

$$V(x;\epsilon) = V_0(x) + \epsilon^2 V_2(x)$$

near phase transition, and determine the corresponding nonlinear solutions.

Perturbation expansion:

$$\psi(x,z) = \left(\epsilon u_1(x,Z) + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots\right) \mathrm{e}^{\mathrm{i}\mu_0 z},$$

where $Z = \epsilon z$.

# 3. Nonlinear dynamics near phase transition

The $u_1$ and $u_2$ solutions are

$$u_1 = A(Z)u_e(x),$$
$$u_2 = -\mathrm{i}A_Z u_g.$$

At order $\epsilon^3$ we have

$$L_0 u_3 = -A_{ZZ}u_g - AV_2 u_e - \sigma|A|^2 A|u_e|^2 u_e.$$

The solvability condition of this equation is

$$A_{ZZ} - \alpha A + \sigma_1 |A|^2 A = 0, \qquad (2)$$

where

$$\alpha = -\frac{\int_{-\infty}^{\infty} V_2 u_e^2 \mathrm{d}x}{\int_{-\infty}^{\infty} u_e u_g \mathrm{d}x}, \quad \sigma_1 = \frac{\sigma \int_{-\infty}^{\infty} |u_e|^2 u_e^2 \mathrm{d}x}{\int_{-\infty}^{\infty} u_e u_g \mathrm{d}x}.$$

Note: this $A$ equation (2) is our reduced model equation for the amplitude of nonlinear solutions, $\alpha > 0$, and $\sigma_1$ is a real constant.

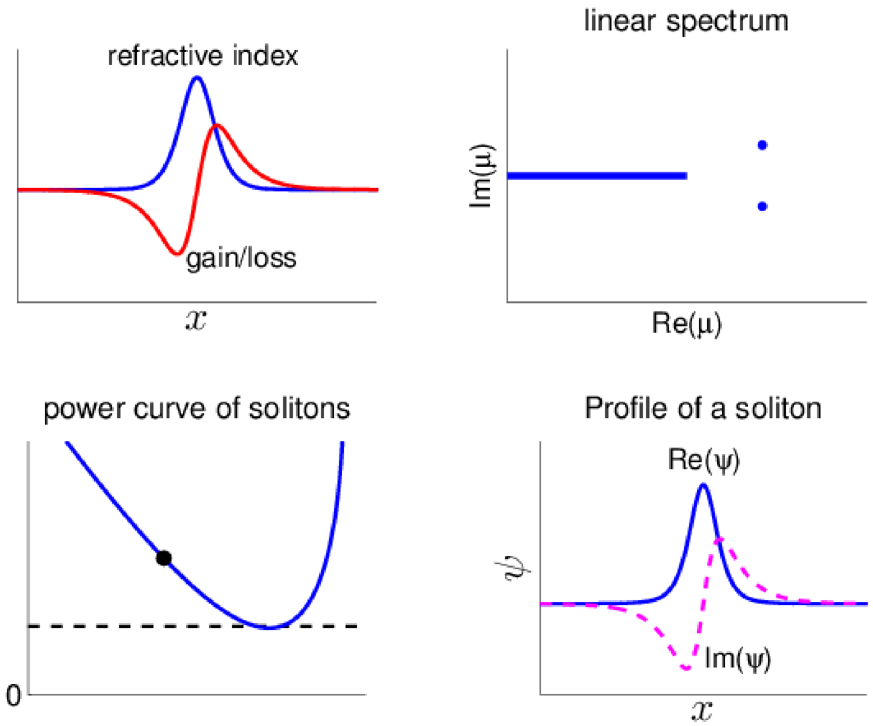
# 3. Nonlinear dynamics near phase transition

Predictions of our model equation:

1. There is a continuous family of soliton solutions (not bifurcating from linear modes) above phase transition.

   Reason: the ODE model admits a family of stationary solutions with non-zero minimum amplitude,

$$A(Z) = A_0 e^{i\mu_1 Z}, \quad \mu_1 = \pm\sqrt{\sigma_1 A_0^2 - \alpha}, \quad |A_0|^2 \geq \frac{\alpha}{|\sigma_1|}.$$



# 3. Nonlinear dynamics near phase transition

**Comment:**

It is often easier to predict soliton families bifurcating from infinitesimal (linear) modes.

It is harder to predict soliton families NOT bifurcating from linear modes.

Our analysis was successful in doing this.

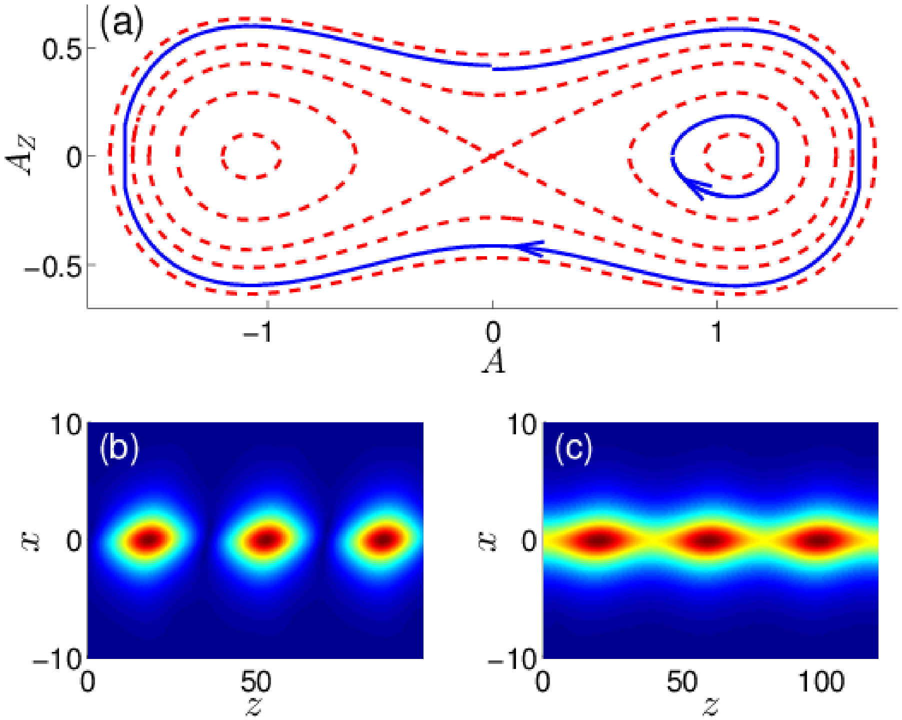
Implications for PT lasers at nonlinear stage:

As the lasing beam amplifies and nonlinearity becomes significant, the lasing beam can approach one of those stable soliton states at higher powers.

# 3. Nonlinear dynamics near phase transition

Predictions of our model equation:

2. Above phase transition, oscillating solutions also exist. Reason:




Implications for PT lasers at nonlinear stage:

As the lasing beam amplifies and nonlinearity becomes significant, the lasing beam may also reach a state with oscillating power levels.

## 4. Symmetry breaking of solitons in PT-symmetric media

A very surprising phenomenon in PT systems is symmetry-breaking bifurcation of solitons.

We find that for a special class of PT-symmetric potentials,

$$V(x) = -\left[g^2(x) + i g'(x)\right],$$

$g(x)$: an **arbitrary** real and even function
$\alpha$: an **arbitrary** real constant,

symmetry breaking can always occur (Yang OL 2014).

# 4. Symmetry breaking of solitons in PT-symmetric media

This is surprising for a number of reasons.

- For asymmetric solitons in a PT-symmetric potential, it is harder to balance gain and loss, especially since these asymmetric solitons exist at arbitrary powers

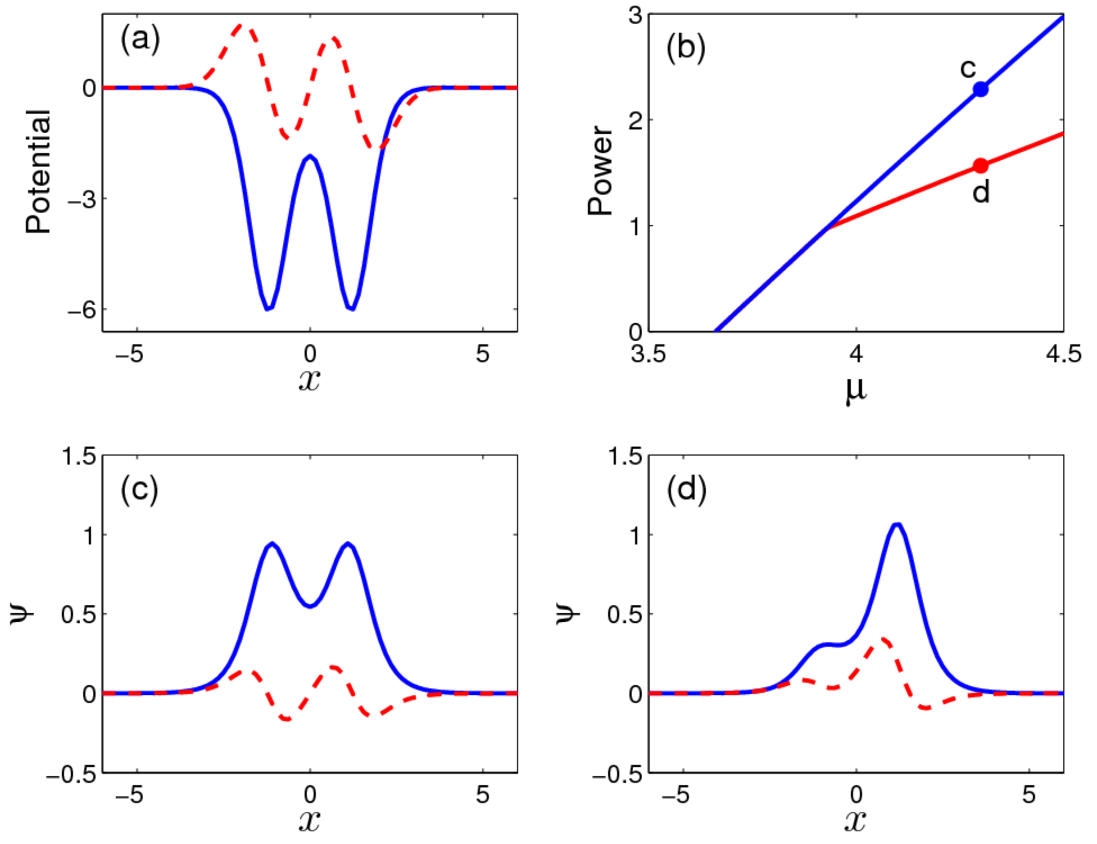
- In a generic PT-symmetric potential, in order for symmetry breaking to occur, infinitely many nontrivial conditions have to be satisfied.

# 4. Symmetry breaking of solitons in PT-symmetric media

Example:

$$V(x) = -\left[g^2(x) + \alpha g(x) + i g'(x)\right],$$

$$g(x) = 2\left(e^{-(x+1.2)^2} + e^{-(x-1.2)^2}\right), \quad \alpha = 1.$$




**What this means is that, in PT lasers, when the laser power becomes very high, its lasing profile may break PT symmetry.**

# 5. Soliton families in non-PT-symmetric media

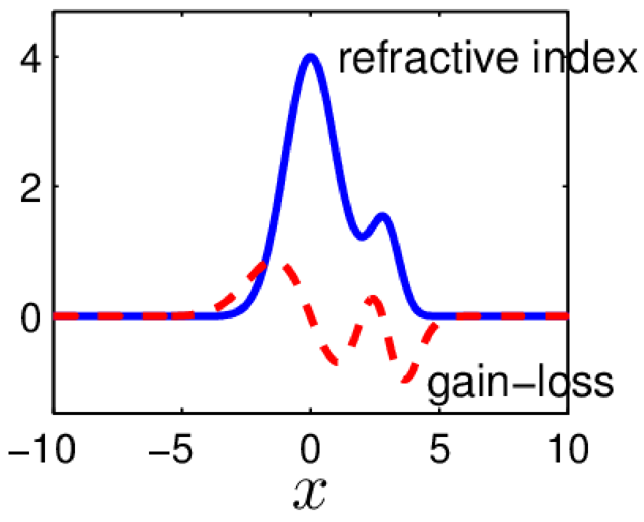
Why does symmetry breaking occur in PT systems?
A challenging question.

One step toward its understanding is to study:

properties of solitons in **non-PT-symmetric potentials** of a similar form

$$V(x) = -\left[g^2(x) + ig'(x)\right],$$

$g(x)$: an arbitrary real **asymmetric** function

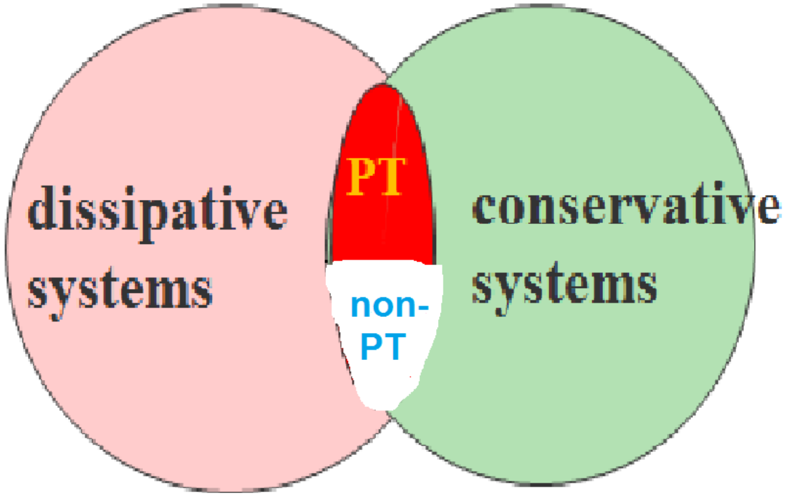


# 5. Soliton families in non-PT-symmetric media

Why study these non-PT-symmetric potentials?

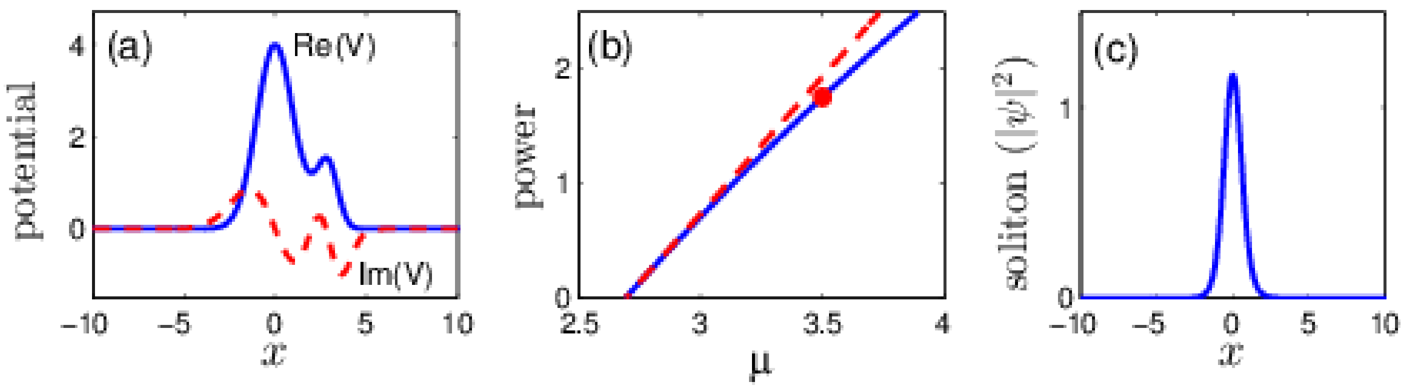
Because

- these potentials are dissipative (with gain and loss)
- they are non-PT-symmetric
- yet we will show that they possess remarkable properties analogous to PT systems and conservative systems



# 5. Soliton families in non-PT-symmetric media

One remarkable property of these non-PT potentials is that they admit continuous families of solitons bifurcating from linear modes, similar to conservative systems.




These soliton families are surprising because there is no PT symmetry here.

How do we understand these soliton families in non-PT potentials?

## 5. Soliton families in non-PT-symmetric media

Mathematical formulation:

$$\mathrm{i}\Psi_z + \Psi_{xx} + [g^2(x) + ig'(x)]\Psi + \sigma|\Psi|^2\Psi = 0.$$

Solitons: $\Psi(x,t) = \psi(x)e^{i\mu z}$, where

$$\psi_{xx} - \mu\psi + [g^2(x) + ig'(x)]\psi + \sigma|\psi|^2\psi = 0. \qquad (3)$$

Question: why does this dissipative non-PT system admit continuous families of solitons bifurcating from linear modes?

This is a nontrivial question, and it is hard to answer using the complex variable $\psi$.

So we will reformulate this problem using polar variables

$$\psi(x) = r(x)e^{i\int\theta(x)dx}$$

## 5. Soliton families in non-PT-symmetric media

One key step in our reformulation is that, the soliton equation (3) admits a constant of motion

$$\frac{dJ}{dx} = 0,$$

where

$$J = r_x^2 - \mu r^2 + \frac{\sigma}{2} r^4 + r^2(\theta + g)^2.$$

Using this constant of motion, the soliton equation for $\psi$ can be reformulated as a new equation for $R = r^2$:

$$R_{xx} - 4\mu R + 3\sigma R^2 \pm 2g\sqrt{4\mu R^2 - 2\sigma R^3 - R_x^2} = 0,$$

This new equation is for a **real** variable $R$ only!

## 5. Soliton families in non-PT-symmetric media

At low amplitude, this new equation becomes

$$R_{xx} - 4\mu R \pm 2g\sqrt{4\mu R^2 - R_x^2} = 0,$$

which is a new type of eigenvalue problem:

- it is scaling-invariant in $R$ but is nonlinear in $R$;
- its eigenvalue $\mu$ also appears in a nonlinear way.

This new eigenvalue problem admits eigenmodes $(R, \mu) = (\phi, \mu_0)$.

From this infinitesimal eigenmode, we can use perturbation expansion to construct its soliton solutions.

But this perturbation calculation is very different from conventional perturbation calculations of solitons (Nixon and Yang, 2015, submitted).

# 5. Soliton families in non-PT-symmetric media

To see why, we do the perturbation expansion

$$R = \epsilon(R_0 + \epsilon R_1 + \epsilon^2 R_2 + \dots),$$
$$\mu = \mu_0 + \epsilon.$$

At order $\epsilon$, we find

$$R_0 = c_0 \phi,$$

where $c_0$ is a positive constant to be determined.

At order $\epsilon^2$, we get

$$LR_1 = F,$$

where

$$L = \partial_{xx} + p_1 \partial_x + p_2,$$

$$F = c_0(f_1 - c_0 \sigma f_2),$$

and $p_1, p_2, f_1, f_2$ are certain functions of eigenmode $\phi$.

# 5. Soliton families in non-PT-symmetric media

What is unusual about this perturbation calculation? Because in the R1 equation

$$LR_1 = F,$$

All functions in the kernel of the adjoint operator L^A are unbounded.

Thus it is not obvious how to apply Fredholm Alternative to derive the solvability condition.

Identifying this solvability condition and applying it to this perturbation calculation is a new feature of the problem.

We have identified this solvability condition and succeeded in finishing this perturbation procedure.

The outcome of this calculation: indeed a continuous family of solitons can bifurcate out from linear modes for non-PT potentials of the form $g^2(x) + ig'(x)$.

# 5. Soliton families in non-PT-symmetric media

We have also shown that in the NLS equation with a non-PT complex potential,

$$\mathrm{i}\Psi_z + \Psi_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0,$$

only potentials of the form

$$V(x) = -\left[g^2(x) + ig'(x)\right],$$

admit continuous families of solitons.

No other potentials share this property.

## 5. Soliton families in non-PT-symmetric media

How do we show this?

Soliton solutions: $\Psi(x,t) = \psi(x)e^{i\mu t}$, where

$$\psi_{xx} - \mu\psi + V(x)\psi + \sigma|\psi|^2\psi = 0.$$

Observation: for soliton families, existence of a constant of motion is critical.

$J(x,\psi)$ is a constant of motion in the soliton equation if $\frac{dJ}{dx} = 0$.

Thus, our strategy is to determine, what complex potentials $V(x)$ admit a constant of motion?

## 5. Soliton families in non-PT-symmetric media

First we split the complex potential $V(x)$ into real and imaginary parts,

$$V(x) = v_1(x) + iv_2(x),$$

where $v_1(x), v_2(x)$ are real functions.

We also express the complex function $\psi(x)$ in polar forms,

$$\psi(x) = r(x)e^{i\int \theta(x)dx},$$

Substituting these expressions into the soliton equation, we get

$$\begin{aligned} r_{xx} - \mu r + v_1 r + \sigma r^3 - \theta^2 r &= 0, \\ (r^2\theta)_x &= -v_2 r^2. \end{aligned}$$

## 5. Soliton families in non-PT-symmetric media

In the absence of the potential ($v_1 = v_2 = 0$), this system admits exactly two constants of motion

$$J_1 = r^2\theta,$$

and

$$J_2 = r_x^2 - \mu r^2 + \frac{\sigma}{2} r^4 + r^2\theta^2,$$

where $dJ_1/dx = dJ_2/dx = 0$.

In the presence of the potential, we calculate $dJ_1/dx$ and $dJ_2/dx$, and derive conditions on $(v_1, v_2)$ so that $dJ_k/dx$ is a total derivative of $x$ (hence a constant of motion ensues).

## 5. Soliton families in non-PT-symmetric media

First we consider $dJ_1/dx$. In the presence of potential, we have

$$dJ_1/dx = -v_2 r^2.$$

In order for $dJ_1/dx$ to be a total derivative,

$$v_2 = 0,$$

i.e., the potential $V(x)$ is real (a conservative system).

This is not what we want since we only consider complex potentials.

## 5. Soliton families in non-PT-symmetric media

Next we consider $dJ_2/dx$. After some algebra, we find that

$$\frac{dJ_2}{dx} = W_x + r^2 v_{1x} + 2(r^2\theta)_x \int v_2 dx,$$

where

$$W = -v_1 r^2 - 2r^2\theta \int v_2 dx.$$

Then utilizing the soliton equation, the above equation becomes

$$\frac{dJ_2}{dx} = W_x + r^2 \left( v_{1x} - 2v_2 \int v_2 dx \right).$$

In order for the right side of the above equation to be a total derivative, the necessary and sufficient condition is

$$v_{1x} = 2v_2 \int v_2 dx.$$

## 5. Soliton families in non-PT-symmetric media

This condition can be rewritten as

$$v_{1x} = \left[ \left( \int v_2 dx \right)^2 \right]_x ,$$

thus

$$v_1 = \left( \int v_2 dx \right)^2 + C,$$

where $C$ is an arbitrary constant which can be set zero.

Finally, denoting

$$g = \int v_2 dx,$$

the potential $V(x)$ which admits a constant of motion then is of

the form

$$V(x) = g^2(x) + ig'(x).$$

Thus soliton families can only exist in this class of complex non-PT potentials.

# 6. Extension to 2D

We find that symmetry breaking of 2D solitons is also possible in a class of complex potentials (Yang 2015)

$$V(x, y) = -\left[g^2(x) + ig'(x) + h(y)\right],$$

$g(x)$: an **arbitrary** real and even function
$h(y)$: an **arbitrary** real function of $y$,
$\alpha$: an **arbitrary** real constant.

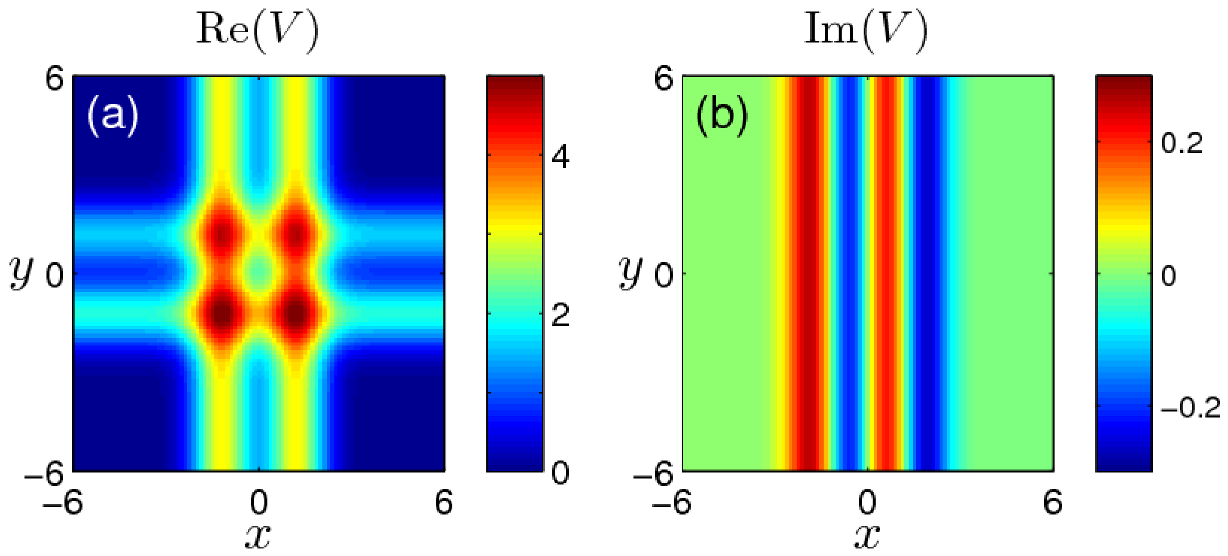
# 6. Extension to 2D

Example:

$$V(x,y) = -\left[g^2(x) + \alpha g(x) + i g'(x) + h(y)\right],$$

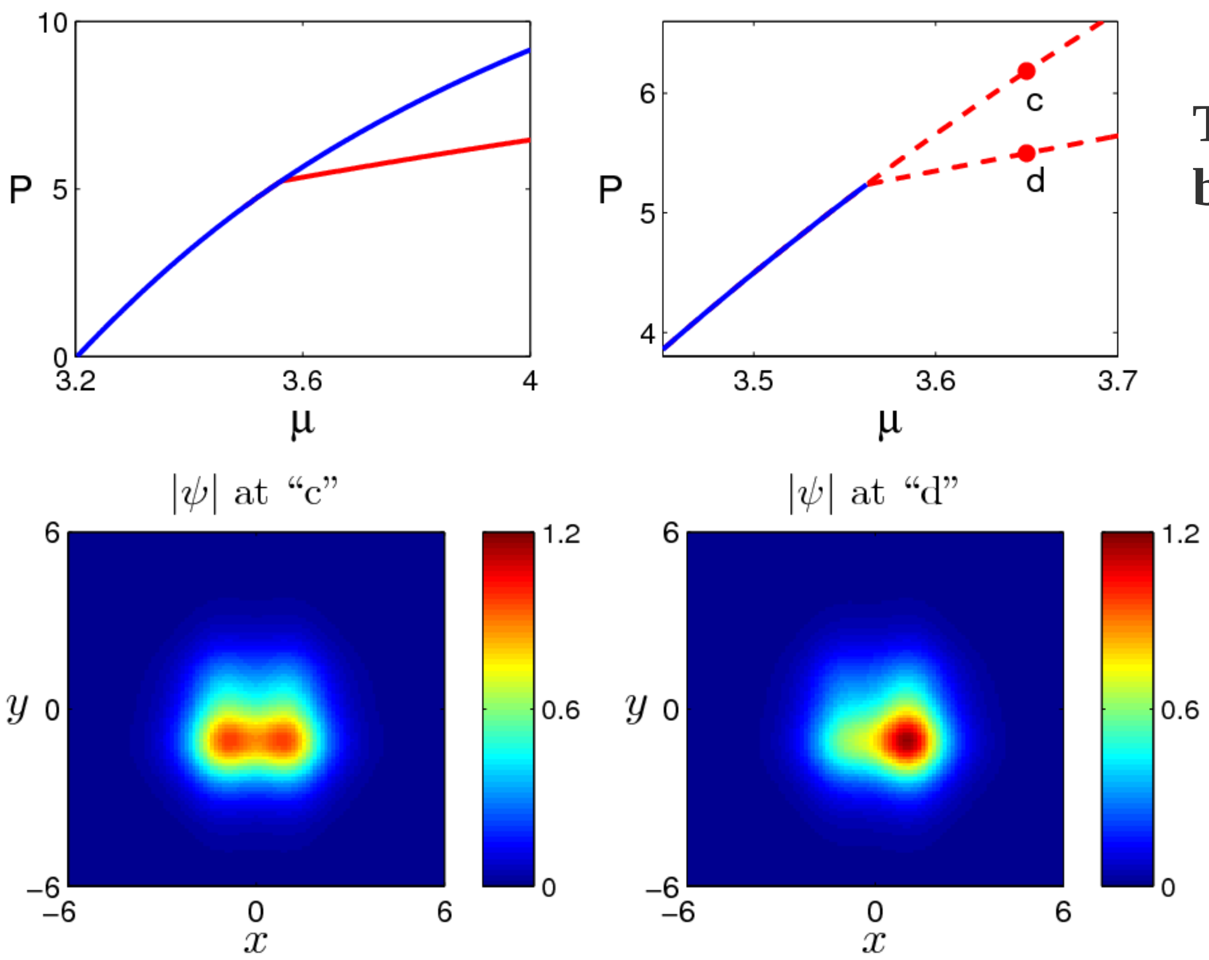
$$g(x) = 0.3\left[e^{-(x+1.2)^2} + e^{-(x-1.2)^2}\right], \quad \alpha = 10,$$

$$h(y) = 2\left[e^{-(y+1.2)^2} + 0.8 e^{-(y-1.2)^2}\right].$$



# 6. Extension to 2D

Symmetry breaking in two dimensions




**The asymmetric branch is unstable**

## 6. Extension to 2D

We have also found that, if potentials of the above form

$$V(x, y) = -\left[g^2(x) + ig'(x) + h(y)\right],$$

are non-PT-symmetric, these potentials admit continuous families of 2D solitons as well.

In other 2D non-PT potentials, soliton families are forbidden, and the wave system behaves as a typical dissipative system.

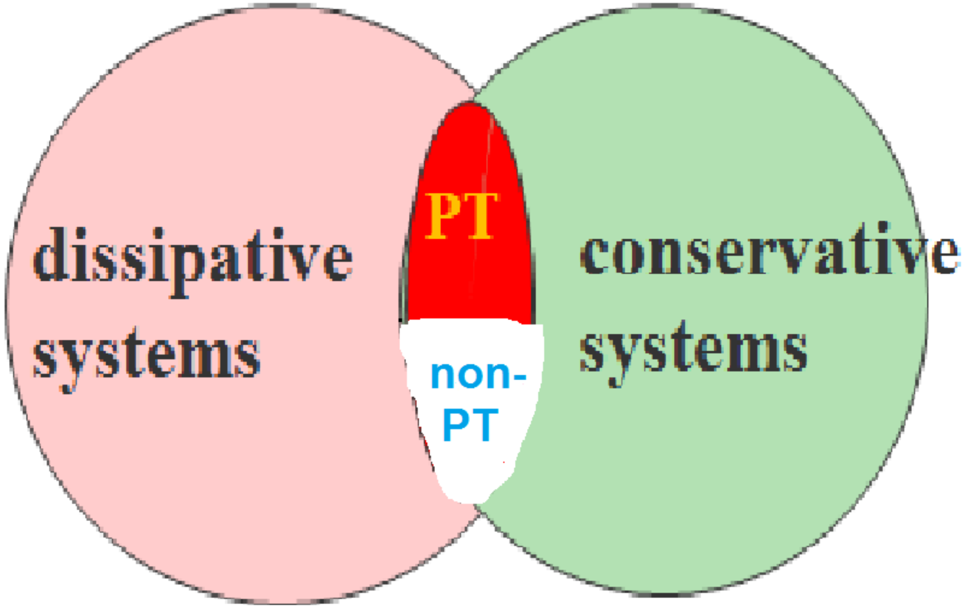
# Technical Summary

- Study of nonlinear effects in PT optics is a research frontier
- In this talk, we studied nonlinear dynamics of PT optics above phase transition, and showed analytically that families of stable solitons and oscillating states prevail under the effects of nonlinearity.

- We also analyzed why in a class of non-PT-symmetric complex potentials, families of solitons exist. This study paves the way for the understanding of symmetry breaking in that same type of potentials.

- These findings are mathematically and physically surprising, and they could have interesting implications for operations of certain PT devices (such as PT lasers).

# Non-technical Summary




We showed that PT-symmetric systems and certain classes of non-PT-symmetric systems, although being dissipative, exhibit properties of conservative systems.

We identified those non-PT-symmetric systems as

- in 1D, $V(x) = -[g^2(x) + ig'(x)]$
- in 2D, $V(x, y) = -\left[g^2(x) + ig'(x) + h(y)\right]$