Long-time asymptotics of Camassa-Holm equations

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- **Joint work with:**
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Outline

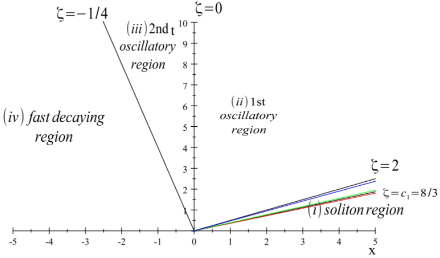
- Historial background of long-time asymptotics–Review of previous works.
- Q: how long it takes for us to get a good approximation between long-time asymptotics and analytical solutions?
  - Construct an initial condition with non-zero reflection coefficients.
  - Compare difference between finite difference and asymptotic solutions.
- Ongoing work: Numerical approach of inverse scattering with the above scattering data.
- Conclusion

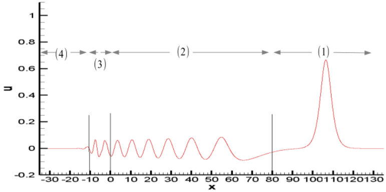
- **Historial review**:
- Zakharov &Manakov (1976), Ablowitz & Segur (1977).
- Segur & Ablowitz (1981): KdV, Painlevé II.
- Deift and Zhou (1990), Deift, Venakides & Zhou (1994), Krüger & Teschl (2009), Grunert & Teschl (2009): KdV.
- Buckingham & Venakides (2007), Boutet de Monvel, Its & Kotlyarov (2007), Boutet de Monvel, Kotlyarov, Shepelsky & Zheng (2010), Boutet de Monvel, Kotlyarov & Shepelsky (2011):
  NLS equation on line or half line.
- Boutet de Monvel et al. (2009, 2010): Camassa-Holm equation.
- Boutet de Monvel & Shepelsky (2013): Degasperis-Procesi equation.
- Boutet de Monvel & Shepelsky (2015): Vakhnenko equation.
  and so on....

- **Long time asymptotics** of Camassa-Holm equation:
- Boutet de Monvel, Kostenko, Shepelsky and Teschl (SIMA, 2009): Suppose $u(x,t)$ is a classical solution of Camassa-Holm (CH) equation

$$u_t + 2u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \; x \in \mathbb{R}, \; t > 0. \qquad \text{(CH)}$$

The asymptotics of solutions $u(x,t)$ can be divided into four regions by considering the associated Riemann-Hilbert problem: $(\zeta = \frac{x}{t})$





- **Long time asymptotics :**
- (i) **Soliton region**: $\frac{x}{t} > 2 + \varepsilon$ $\forall$ small $\varepsilon > 0$.
  $u(x, t) = N\text{-soliton} + O\left(t^{-L}\right)$ for any $L > 0$,
- (ii) **First oscillatory region**: $0 \leq \frac{x}{t} > 2 - \varepsilon$
  $u(x, t)$ close to the form of slowly decaying modulated oscillations:

$$u(x, t) = \frac{c_1}{\sqrt{t}} \sin\left(c_2 t + c_3 \log t + c_4\right) + O\left(t^{-\alpha}\right)$$

  for any $\alpha \in \left(\frac{1}{2}, 1\right)$ provided $L \geq 5$, where c_m $(m = 1, \ldots, 4)$ are functions of $\frac{x}{t}$, which are expressed in terms of scattering data.
- (iii) **Second oscillatory region**: $\frac{-1}{4} + \varepsilon < \frac{x}{t} < 0$.
  $u(x, t)$ close to the form of a sum of two slowly decaying modulated oscillations.
- (iv) **Fast decay region**: $\frac{x}{t} < \frac{-1}{4} - \varepsilon$.
  $u(x, t) \sim 0$.

- Boutet de Monvel, Its and Shepelsky (SIMA, 2010):
- Continuing the results in 2009, in the two **transition** regions:
  (T1): the region between (i) and (ii),
  (T2): the region between (iii) and (iv),
  the asymptotics of solutions is expressed by solutions of Painlevé II equation.

$$w''(x) = 2w^3(x) + xw(x). \tag{P2}$$

- **(T1) region**: For $\left|\frac{x}{t}-2\right| t^{\frac{2}{3}} < \varepsilon$ with any $\varepsilon > 0$,

$$u(x,t) = -(4/3)^{\frac{2}{3}} t^{-\frac{2}{3}} \left(w_1^2(z) - w_1'(z) \right) + O\left(t^{-1}\right),$$

  where $z = 6^{\frac{-1}{3}} \left(\frac{x}{t}-2\right) t^{\frac{2}{3}}$,

- **(T2) region**: For $\left|\frac{x}{t}+\frac{1}{4}\right| t^{\frac{2}{3}} < \varepsilon$ with any $\varepsilon > 0$,

$$u(x,t) = 12^{\frac{1}{6}} t^{-\frac{1}{3}} w_2(y) \sin\left(\frac{-3\sqrt{3}}{4} t - \frac{3^{\frac{5}{6}}}{2^{\frac{4}{3}}} y t^{\frac{1}{3}} + \Delta \right) + O\left(t^{-\frac{2}{3}}\right),$$

  where $y = -\left(\frac{16}{3}\right)^{\frac{1}{3}} \left(\frac{x}{t}+\frac{1}{4}\right) t^{\frac{2}{3}}$, Δ being a function, depends on scattering data.
  $w_1(z)$ & $w_2(y)$: the real-valued, non-singular solution of (P2),
  $w_1(z) \sim -R(0)\mathrm{Ai}(z)$ as $z \to \infty$,
  $w_2(y) \sim \left|R\left(\sqrt{3}/2\right)\right| \mathrm{Ai}(y)$ as $y \to \infty$,
  where $R(k)$: right reflection coefficients.

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- Then we compare the finite difference and asymptotic solutions to partially answer this question.
- The asymptotic solutions in (i) to (T2) depend on $u(x,0)$.
- Given an arbitrary $u(x,0)$, usually we can't express asymptotic solution explicitly.
- We need a proper (or specific) $u(x,0)$ such that the asymptotic solution can be calculated.

- Boutet de Monvel, Kotlyarov, Shepelsky, Zheng (2010): Initial boundary value problems of nonlinear Schrödinger equation $iu_t + u_{xx} + 2|u|^2 u = 0$.
- They use some initial conditions to obtain numerical solutions indicating that the long-time asymptotics seems to appear.
- They didn't consider comparison between asymptotic and numerical solutions.

Result 1

Initial condition of Camassa-Holm equation with non-zero reflection coefficients

- CH equation $u_t + 2u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
- Let the momentum $m(x,t) := u(x,t) - u_{xx}(x,t) + 1$.
- Cauchy problem for CH equation:
- We find the initial condition $u(x,0)$ s.t.

  (i) $u(x,0)$: smooth, rapidly decreasing as $|x| \to \infty$,
  (ii) $m(x,0) > 0$, (then $m(x,t) > 0 \ \forall t > 0$)
  (iii) $\int_{\mathbb{R}} (1+|x|)^{1+n} (|m(x,0) - 1| + |m_x(x,0)| + |m_{xx}(x,0)|)\, dx < \infty$ for some $n \in \mathbb{N}$.

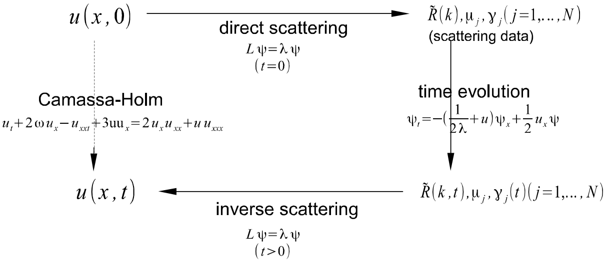
- Existence of classical solutions: Constantin and Escher (1998).

- The Lax pair of CH (Camassa and Holm (1993)) is

$$L\psi := \frac{1}{m}\left(-\psi_{xx} + \frac{1}{4}\psi\right) = \lambda\psi, \qquad \text{(Lax1)}$$

$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi. \qquad \text{(Lax2)}$$

- $(\psi_t)_{xx} = (\psi_{xx})_t$ iff CH equation holds.



$$L\psi = \frac{1}{m(x,t)}\left(-\psi_{xx} + \frac{1}{4}\psi\right) = \lambda\psi \qquad \text{(Lax1)}$$

- Let $\lambda = \frac{1}{4} + k^2$,

$$y = x - \int_x^\infty \left(\sqrt{m(r,t)} - 1\right) dr, \text{ hence } \frac{dy}{dx} = \sqrt{m(x,t)}$$

$$\tilde{\psi}(y) = (m(x,t))^{\frac{1}{4}} \psi(x),$$

then (Lax1) can be transformed to

$$-\tilde{\psi}_{yy} + q(y,t)\tilde{\psi} = k^2\tilde{\psi}$$

with

$$q(y,t) = \frac{m_{yy}(y,t)}{4m(y,t)} - \frac{3}{16}\frac{(m_y)^2(y,t)}{m^2(y,t)} + \frac{1-m(y,t)}{4m(y,t)}.$$

- By the integral condition of $m(x,0)$,

$$\int_{-\infty}^{\infty} (1+|y|)^{1+n} |q(y,0)|\, dy < \infty.$$

- Faddeev (1958), Deift & Trubowitz (1979), Marchenko (1986):
- (1) Discrete spectrum ($k^2 < 0$):

eigenvalues: $k = i\mu_j$, $j = 1, ..., N$ for some $N \in \mathbb{N}$, with the corresponding eigenfunction: $\tilde{\psi}_j$,
Let γ_j^+ (right norming constants) be defined by

$$\tilde{\psi}_j(y) = \gamma_j^+ e^{-\mu_j y} + o(1) \text{ as } y \to \infty,$$

and γ_j^- (left norming constants) be defined by

$$\tilde{\psi}_j(y) = \gamma_j^- e^{\mu_j y} + o(1) \text{ as } y \to -\infty.$$

- (2) Continuous spectrum:
  Let $\hat{\psi} = \hat{\psi}(y, k)$ be the continuous eigenfunction for each $k \in \mathbb{R}$,

$$\hat{\psi} \sim \begin{cases} e^{-iky} + R_+(k)e^{iky}; \text{ as } y \to \infty, \\ T(k)\, e^{-iky}; \text{ as } y \to -\infty \end{cases}$$

  where $T(k)$: transmission coefficient,
  $R_+(k)$: **right** reflection coefficient.

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$$\hat{\psi} \sim \begin{cases} e^{iky} + R_-(k)e^{-iky}; \text{ as } y \to -\infty, \\ T(k)\, e^{iky}; \text{ as } y \to \infty. \end{cases}$$

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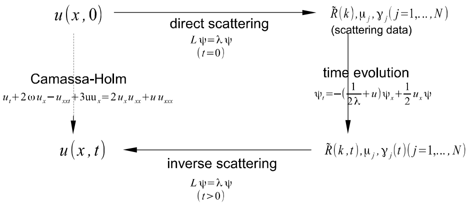
- Scattering data: $R_+(k), \left\{\mu_j, \gamma_j^+\right\}_{j=1}^N$ or $R_-(k), \left\{\mu_j, \gamma_j^-\right\}_{j=1}^N$ w.r.t. $q(y, 0)$.



- Direct problem:

$$u(x,0) \to m(x,0) \to m(y,0) \to q(y,0) \to \text{scattering data at } t = 0.$$

- Inverse problem:

$$u(x,t) \leftarrow m(x,t) \leftarrow m(y,t) \leftarrow q(y,t) \leftarrow \text{scattering data at } t > 0.$$

- The processes $m(x,t) \leftarrow m(y,t) \leftarrow q(y,t)$ is the main difference between the IST of CH and KdV.

- How does the scattering data influence the long-time asymptotics?
- By the long-time asymptotic results, we have:
  - The number of discrete eigenvalues μ_j $(j = 1, ..., N)$ determines the number of solitary waves in the soliton region (i).
  - The appeareance of reflection coefficient $R_{\pm}(k)$ makes the oscillatory phenomenon in oscillatory regions (ii) and (iii) to occur.
    If $R_{\pm}(k) \equiv 0 \Rightarrow$ pure soliton.
  - Ex: in region (ii),

$$u(x,t) \sim \frac{c_1}{\sqrt{t}} \sin\left(c_2 t + c_3 \log t + c_4\right),$$

$$c_1 = -c_5 \sqrt{\frac{-1}{2\pi} \log\left(1 - R_+\left(c_6\right)\right)^2}.$$

- Inverse scattering of CH equations:
- Constantin (2001): continuous spectrum,
- Johnson (2002), Lenells (2002),
  Constantin and Lenells (2003)
  Li and Zhang (2004),
  Constantin, Gerdjikov & Ivanov (2007) et al...

  Inverse scattering with zero reflection coefficients to construct the pure N- soliton solutions.

- **Theorem (**C.- and Sheu (JNMP 2015)**)**:
  Let $0 < q_0 < 1$, consider the CH equation subject to the following initial condition

$$u_{q_0}(x,0) = \begin{cases} \frac{A(A+1+\log(e^x - A))}{e^x}, & \text{for } x \geq \log(1+A), \\ \frac{A\left(A+1+\log\left((1+A)^2 e^{-x} - A\right)\right)}{(1+A)^2 e^{-x}}, & \text{for } x < \log(1+A). \end{cases}$$

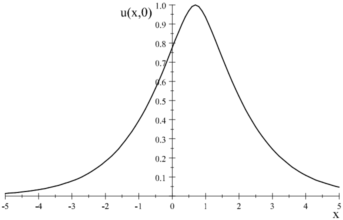
  where $A := \frac{q_0}{1-q_0}$. The above initial condition in space-time domain corresponds to the following scattering data in spectral domain:

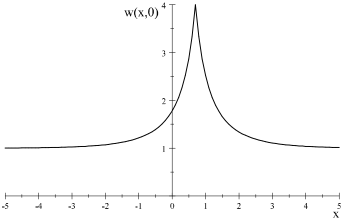
$$R_{\pm}(k) = \frac{-q_0}{q_0 + 2ik}, \quad \mu_1 = \frac{q_0}{2}, \quad \gamma_1^{\pm} = \sqrt{\frac{q_0}{2}}.$$

- Momentum $m(x,t) := u(x,t) - u_{xx}(x,t) + 1$.
- Initial momeutum

$$m(x,0) = \begin{cases} \left(\frac{1}{1-Ae^{-x}}\right)^2, & \text{for } x \geq \ln(1+A), \\ \left(\frac{(1+A)^2}{(1+A)^2 - Ae^{x}}\right)^2, & \text{for } x < \ln(1+A). \end{cases}$$

$m(x,0) > 0$.
- Consider $q_0 = \frac{1}{2}$:





- Proof:
- **Lemma** (Whitham (1974), Drazin & Johnson (1989))
  If $q(y) = -q_0 \delta(y)$ then the scattering data corresponding to

$$-\tilde{\psi}_{yy} + q(y)\tilde{\psi} = k^2 \tilde{\psi}$$

  is

$$R_{\pm}(k) = \frac{-q_0}{q_0 + 2ik}, \quad \mu_1 = \frac{q_0}{2}, \quad \gamma_1^{\pm} = \sqrt{\frac{q_0}{2}}.$$

- We recover $m(y,0)$ from $q(y,0) = -q_0 \delta(y)$
  (inverse scattering at $t = 0$).

- (Continued)
- Constantin's version of IST to recover $m(y,t)$ from $q(y,t)$ (2001). Find the positive solution $C = C(y,t)$ from

$$C_{yy} = C\left(q(y,t) + \frac{1}{4}\right) - \frac{1}{4C^3}, \quad \lim_{|y|\to\infty} C(y,t) = 1.$$

then $m(y,t) = C^4(y,t)$ (put $q(y,0) = -q_0\delta(y)$).
- Find $u(y,0)$ from $m(y,0)$: change $u - u_{xx} = m - 1$ to

$$mu_{yy} + \frac{1}{2}m_y u_y - u = 1 - m$$

by $\frac{dy}{dx} = m(y,t)^{\frac{1}{2}}$.
- Find change of variable between y and x by solving

$$\frac{dy}{dx} = m(y,0)^{\frac{1}{2}}, \quad \lim_{x\to\infty}(y(x) - x) = 0.$$

Result 2:

Difference between finite difference and asymptotic solutions

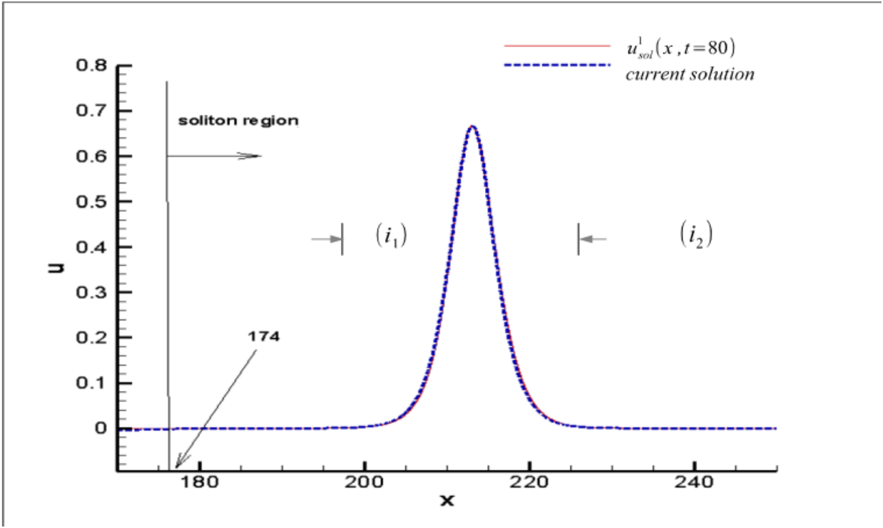
- **Theorem (**C.- Yu and Sheu (arXiv: 1412.1234)**)**:
  Let $0 < q_0 < 1$, consider the CH equation subject to the initial condition $u_{q_0}(x, 0)$. Then the long time needed to approximate the asymptotic solutions in the six regions (i)-(iv), (T1) and (T2) can be found.

- **Theorem (**C.- Yu and Sheu (arXiv: 1412.1234)**)**:
  Let $0 < q_0 < 1$, consider the CH equation subject to the initial condition $u_{q_0}(x, 0)$. Then the long time needed to approximate the asymptotic solutions in the six regions (i)-(iv), (T1) and (T2) can be found.
- The long-time asymptotic solutions depend on the scattering data (Boutet de Monvel et al.'s results).
- Since now under $u_{q_0}(x, 0)$, the scattering data can be explicitly expressed, the asymptotic solution can be written down.

- (i) soliton region: $\frac{x}{t} > 2 + \varepsilon$

$$\begin{cases} u(y,t) = \frac{4q_0^2}{\left(1-q_0^2\right)^2} \frac{1}{\exp\left(q_0\left(y-\frac{2}{1-q_0^2}t\right)\right)+\frac{1}{4}\exp\left(-q_0\left(y-\frac{2}{1-q_0^2}t\right)\right)+\frac{1+q_0^2}{1-q_0^2}}, \\ x(y,t) = y + \log \frac{1+\exp\left(-q_0\left(y-\frac{2}{1-q_0^2}t+\frac{1}{q_0}\log 2\right)\right)\frac{1+q_0}{1-q_0}}{1+\exp\left(-q_0\left(y-\frac{2}{1-q_0^2}t+\frac{1}{q_0}\log 2\right)\right)\frac{1-q_0}{1+q_0}}. \end{cases}$$

- Take $\varepsilon = \frac{14}{80}$, $t = 80$:



- (ii) first oscillatory region: $0 \leq \zeta := \frac{x}{t} < 2 - \varepsilon$:

$$u(x,t) = \frac{c_1^{(0)}(\zeta)}{\sqrt{t}} \sin\left(\frac{2k_0^3(\zeta)}{\left(\frac{1}{4}+k_0^2(\zeta)\right)^2} t - \nu_0(\zeta) \log t + \delta_0(\zeta) \right) + O\left(t^{-\alpha}\right)$$

where

$$\nu_0(\zeta) = -\frac{1}{2\pi} \log \frac{\sqrt{1+4\zeta}-1-\zeta}{\zeta q_0^2 + \sqrt{1+4\zeta}-1-\zeta}, \quad k_0(\zeta) = \frac{1}{2}\sqrt{-\frac{1+\zeta-\sqrt{1+4\zeta}}{\zeta}},$$

$$c_1^{(0)}(\zeta) = -\sqrt{\frac{2k_0(\zeta)\nu_0(\zeta)}{\left(\frac{1}{4}+k_0^2(\zeta)\right)\left(\frac{3}{4}-k_0^2(\zeta)\right)}}$$

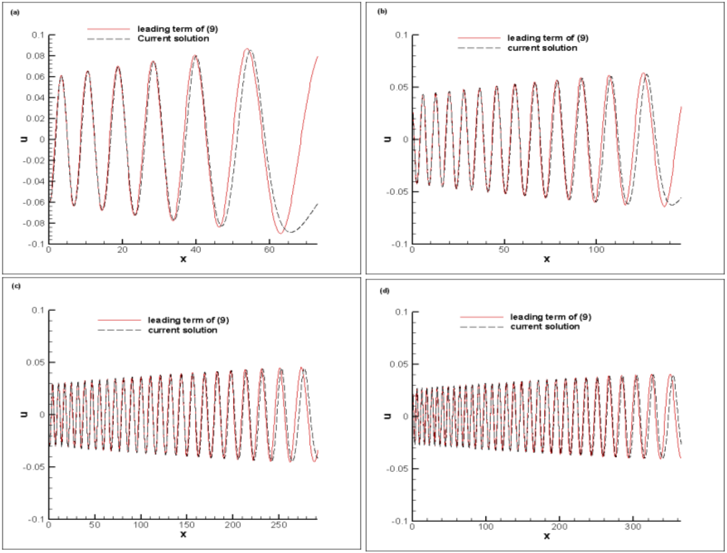
and

$$
\begin{aligned}
\delta_0(\zeta) &= \frac{\pi}{4} - \left(\tan^{-1}\left(\frac{-2k_0(\zeta)}{q_0}\right) + \pi\right) \\
&-\nu_0(\zeta)\log\frac{32\zeta\left(\sqrt{1+4\zeta}-1-\zeta\right)\left(1+4\zeta-\sqrt{1+4\zeta}\right)}{\left(\sqrt{1+4\zeta}-1\right)^3} \\
&+4\tan^{-1}\left(\frac{q_0}{2k_0(\zeta)}\right) + 4k_0(\zeta)\log\frac{1+q_0}{1-q_0} \\
&+\frac{4k_0(\zeta)}{\pi}\int_{-k_0(\zeta)}^{k_0(\zeta)}\frac{1}{1+4\xi^2}\log\frac{4\xi^2}{q_0^2+4\xi^2}d\xi \\
&+\frac{1}{\pi}\int_{-k_0(\zeta)}^{k_0(\zeta)}\frac{2q_0^2}{\left(q_0^2+4\xi^2\right)\xi}\log\left(k_0(\zeta)-\xi\right)d\xi + \arg\Gamma\left(i\nu_0(\zeta)\right).
\end{aligned}
$$

$$
\begin{aligned}
\arg\Gamma\left(i\nu_0(\zeta)\right) &= \arg\Gamma\left(1+i\nu_0(\zeta)\right) - \frac{\pi}{2} \\
&= \nu_0\frac{\Gamma'(1)}{\Gamma(1)} + \sum_{n=0}^{\infty}\left(\frac{\nu_0}{1+n} - \tan^{-1}\frac{\nu_0}{1+n}\right) - \frac{\pi}{2} \\
&= \left(-\gamma\nu_0(\zeta) + \sum_{n=0}^{\infty}\left(\frac{\nu_0(\zeta)}{1+n} - \tan^{-1}\frac{\nu_0(\zeta)}{1+n}\right)\right) - \frac{\pi}{2}
\end{aligned}
$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.5772156649....$ is the Euler constant.

- Take $\varepsilon = \frac{14}{80}$, at $t = 40, 80, 160, 200$:



- Recall Boutet de Monvel et al.'s results:
  Decay rates:

region (i)	region (ii)
t^{-L}	$t^{-\alpha}$, $\alpha \in \left(\frac{1}{2}, 1\right)$ provided $L \geq 5$

- We need a longer time to get the asymptotic solution in region (ii) than region (i).

- (iii) second oscillatory region: $\frac{-1}{4} + \varepsilon \leq \frac{x}{t} < 0$:

$$
\begin{aligned}
u(x,t) = \quad & \frac{c_1^{(0)}(\zeta)}{\sqrt{t}} \sin\left(\frac{2k_0^3(\zeta)}{\left(\frac{1}{4}+k_0^2(\zeta)\right)^2} t - \nu_0(\zeta) \log t + \bar{\delta}_0(\zeta) \right) \\
& + \frac{c_1^{(1)}(\zeta)}{\sqrt{t}} \sin\left(\frac{2k_1^3(\zeta)}{\left(\frac{1}{4}+k_1^2(\zeta)\right)^2} t + \nu_1(\zeta) \log t - \delta_1(\zeta) \right) \\
& + O\left(t^{-\alpha}\right)
\end{aligned}
$$

where

$$
k_0(\zeta) = \frac{1}{2}\sqrt{-\frac{1+\zeta-\sqrt{1+4\zeta}}{\zeta}}, \quad k_1(\zeta) = \frac{1}{2}\sqrt{-\frac{1+\zeta+\sqrt{1+4\zeta}}{\zeta}},
$$

$$
c_1^{(0)}(\zeta) = -\sqrt{\frac{2k_0(\zeta)\nu_0(\zeta)}{\left(\frac{1}{4}+k_0^2(\zeta)\right)\left(\frac{3}{4}-k_0^2(\zeta)\right)}}, \quad c_1^{(1)}(\zeta) = -\sqrt{\frac{2k_1(\zeta)\nu_1(\zeta)}{\left(\frac{1}{4}+k_1^2(\zeta)\right)\left(k_1^2(\zeta)-\frac{3}{4}\right)}}
$$

$$\nu_0(\zeta) = -\frac{1}{2\pi}\log\frac{\sqrt{1+4\zeta}-1-\zeta}{\zeta q_0^2+\sqrt{1+4\zeta}-1-\zeta}, \quad \nu_1(\zeta) = -\frac{1}{2\pi}\log\frac{-\sqrt{1+4\zeta}-1-\zeta}{\zeta q_0^2-\sqrt{1+4\zeta}-1-\zeta}.$$

$$\begin{aligned}
\bar{\delta}_0(\zeta) &= \frac{\pi}{4} - \left(\tan^{-1}\left(\frac{-2k_0(\zeta)}{q_0}\right)+\pi\right) + \arg\Gamma\left(i\nu_0(\zeta)\right) \\
&-\nu_0(\zeta)\log\frac{32\zeta\left(\sqrt{1+4\zeta}-1-\zeta\right)\left(1+4\zeta-\sqrt{1+4\zeta}\right)}{\left(\sqrt{1+4\zeta}-1\right)^3} \\
&+4\tan^{-1}\left(\frac{q_0}{2k_0(\zeta)}\right) + 4k_0(\zeta)\log\frac{1+q_0}{1-q_0} \\
&+\frac{4k_0(\zeta)}{\pi}\left(\int_{-\infty}^{-k_1(\zeta)} + \int_{-k_0(\zeta)}^{k_0(\zeta)} + \int_{k_1(\zeta)}^{\infty}\right)\frac{1}{1+4\xi^2}\log\frac{4\xi^2}{q_0^2+4\xi^2}d\xi \\
&+\frac{1}{\pi}I_0 + 2\nu_1(\zeta)\log\frac{k_1(\zeta)-k_0(\zeta)}{k_1(\zeta)+k_0(\zeta)}
\end{aligned}$$

$$\begin{aligned}
I_0 = \; & \left(\int_{-\infty}^{-k_1(\zeta)} + \int_{-k_0(\zeta)}^{k_0(\zeta)}\right)\log\left(k_0(\zeta)-\xi\right)\frac{2q_0^2}{\left(q_0^2+4\xi^2\right)\xi}d\xi \\
&+\int_{k_1(\zeta)}^{\infty}\log\left(\xi-k_0(\zeta)\right)\frac{2q_0^2}{\left(q_0^2+4\xi^2\right)\xi}d\xi,
\end{aligned}$$

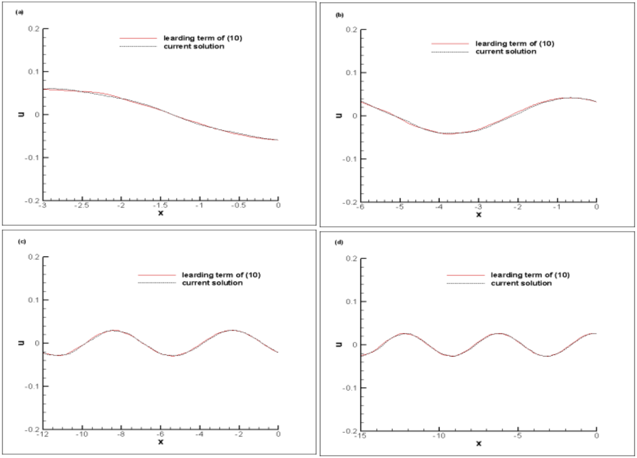
with

$$\begin{aligned}\delta_1(\zeta) &= \frac{\pi}{4} + \tan^{-1}\left(\frac{-2k_1(\zeta)}{q_0}\right) + \pi + \arg\Gamma\left(i\nu_1(\zeta)\right) \\ &-\nu_1(\zeta)\log\frac{32\zeta\left(\sqrt{1+4\zeta}+1+\zeta\right)\left(1+4\zeta+\sqrt{1+4\zeta}\right)}{-\left(\sqrt{1+4\zeta}+1\right)^3} \\ &-4\tan^{-1}\left(\frac{q_0}{2k_1(\zeta)}\right) + 4k_1(\zeta)\log\frac{1+q_0}{1-q_0} \\ &+\frac{4k_1(\zeta)}{\pi}\left(\int_{-\infty}^{-k_1(\zeta)} + \int_{-k_0(\zeta)}^{k_0(\zeta)} + \int_{k_1(\zeta)}^{\infty}\right)\frac{1}{1+4\xi^2}\log\frac{4\xi^2}{q_0^2+4\xi^2}d\xi \\ &-\frac{1}{\pi}I_1 - 2\nu_0(\zeta)\log\frac{k_1(\zeta)-k_0(\zeta)}{k_1(\zeta)+k_0(\zeta)}.\end{aligned}$$

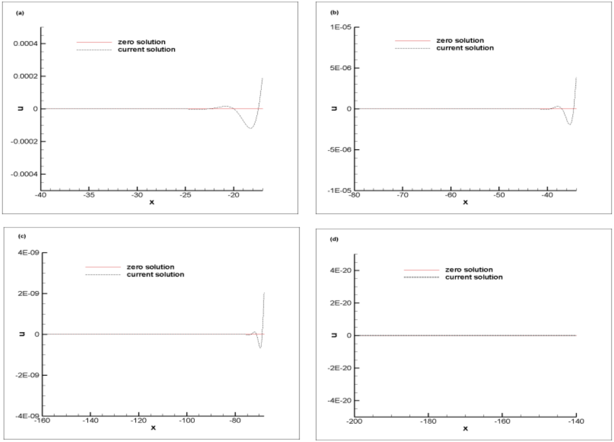
$$\begin{aligned}I_1 = &\left(\int_{-\infty}^{-k_1(\zeta)} + \int_{-k_0(\zeta)}^{k_0(\zeta)}\right)\log\left(k_1(\zeta)-\xi\right)\frac{2q_0^2}{\left(q_0^2+4\xi^2\right)\xi}d\xi \\ &+\int_{k_1(\zeta)}^{\infty}\log\left(\xi-k_1(\zeta)\right)\frac{2q_0^2}{\left(q_0^2+4\xi^2\right)\xi}d\xi,\end{aligned}$$

$$\arg\Gamma\left(i\nu_1(\zeta)\right) = -\gamma\nu_1(\zeta) + \sum_{n=0}^{\infty}\left(\frac{\nu_1(\zeta)}{1+n} - \tan^{-1}\frac{\nu_1(\zeta)}{1+n}\right) - \frac{\pi}{2}.$$

- Take $\varepsilon = \frac{14}{80}$, at $t = 40, 80, 160, 200$:



- (iv) fast decaying region: $\frac{x}{t} < \frac{-1}{4} - \varepsilon$
- Take $\varepsilon = \frac{14}{80}$, at $t = 40, 80, 160, 200$:



- Transition regions:
- Painlevé II equation:

$$w''(z) = 2w^3(z) + zw(z), \ z \in \mathbb{R},$$

with the specified boundary condition $w(z; r) \sim r\mathrm{Ai}(z)$ as $z \to \infty$, where $\mathrm{Ai}(z)$: Airy function.

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- If $|r| = 1$, $w(z;r) \sim \mathrm{sgn}(r)\sqrt{\frac{-1}{2}z}$ as $z \to -\infty$;
- If $|r| < 1$, $w(z;r) \sim d\,|z|^{\frac{-1}{4}} \sin\left\{\frac{2}{3}|z|^{\frac{3}{2}} - \frac{3}{4}d^2 \ln|z| - \theta\right\}$ as $z \to -\infty$,
  $d^2 = \frac{-1}{\pi}\ln\left(1 - r^2\right); \quad \theta = \frac{3}{2}d^2 \ln 2 + \arg\left[\Gamma\left(1 - \frac{i}{2}d^2\right)\right] - \frac{\pi}{4}.$
- If $|r| > 1$, $w(z;r) \sim \mathrm{sgn}(r)\,(z - z_0)^{-1}$ as $z \downarrow z_0$.
- Ablowitz and Segur (1977), Hastings & Mcleod (1980), Clarkson & Mcleod (1988).
- In CH equation, r depends on right reflection coeff.

- In (T1), $r=1$, $w_1(z) \sim \mathrm{Ai}(z)$ as $z \longrightarrow \infty$; $w_1(z) \sim \sqrt{\frac{-1}{2} z}$ as $z \longrightarrow -\infty$;
- Choosing an interval (z_1, z_2), solve (P2) with the boundary condition $w(z_1) = \sqrt{\frac{-1}{2} z_1}$, $w(z_2) = \mathrm{Ai}(z_2) = \frac{1}{2\sqrt{\pi}} (z_2)^{-1/4} e^{-2/3 (z_2)^{3/2}}$. We call this solution as $w_1^{\mathrm{num}}(z)$.

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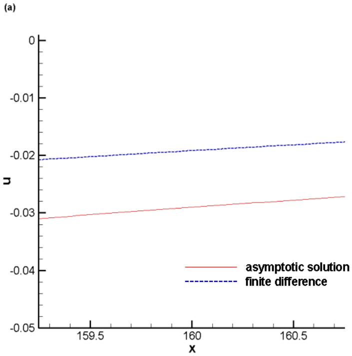
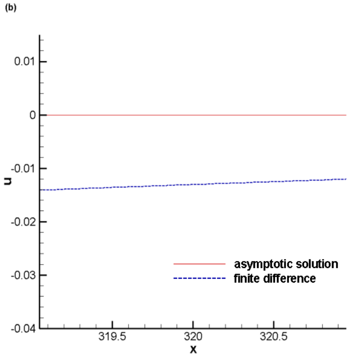
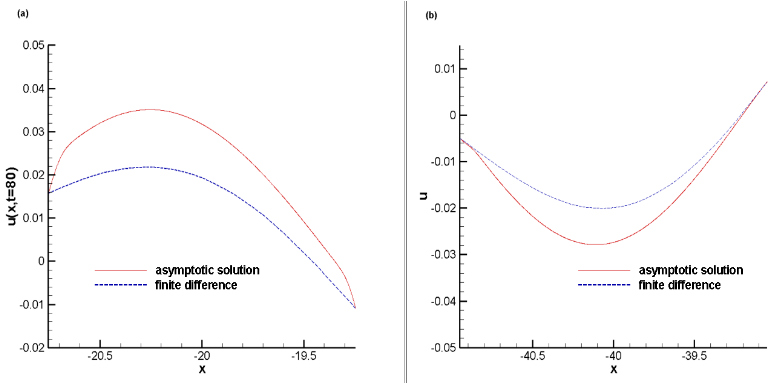
- Take $\varepsilon = \frac{14}{80}$, consider numerical solution and the leading term in the asymptotics in (T1) region with w_1 being replaced by w_1^{num} at $t = 80$ (then $159.25 < x < 160.75$) & $t = 160$ (then $320.95 < x < 319.05$).





- In (T2): $w_2(y) \sim \frac{q_0}{\sqrt{q_0^2+3}} \mathrm{Ai}(y)$ as $y \to \infty$,
  $w_2(y) \sim d\,|z|^{\frac{-1}{4}} \sin\left\{ \frac{2}{3}|y|^{\frac{3}{2}} - \frac{3}{4} d^2 \ln|y| - \theta \right\}$ as $y \to -\infty$;
- Take $\varepsilon = \frac{14}{80}$, $t = 80$ (then $-20.754 < x < -19.246$) & $t = 160$ (then $-40.95 < x < -39.05$):



Ongoing work

Numerical approach of

inverse scattering transform

- **Inverse scattering problem**:

$$L\psi := \frac{1}{m}\left(-\psi_{xx} + \frac{1}{4}\psi\right) = \lambda\psi, \qquad \text{(Lax1)}$$

$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi. \qquad \text{(Lax2)}$$

- The time evolution of $R_{\pm}(k)$, μ_j, $\gamma_j^{\pm}$ can be derived by (Lax2) (Constanin (2001)):

$$\begin{aligned} &\mu_i(t) = \mu_i, \\ &\gamma_j^{\pm}(t) = \gamma_j^{\pm}\exp\left(\frac{\pm\mu_j t}{2(\frac{1}{4}-\mu_j^2)}\right), \;\; i = 1, ..., N, \\ &R_{\pm}(k,t) = R_{\pm}(k)\exp\left(\frac{\mp ikt}{\frac{1}{4}+k^2}\right). \end{aligned}$$

- Gel'fand-Levitan-Marchenko (GLM) integral equation (**right** reflection coefficients):

$$K(y,r,t) + f(y+r,t) + \int_y^\infty K(y,z,t)\, f(z+r,t)\, dz = 0, \;\; y < r$$

$$f(z,t) := \sum_{j=1}^{N} \left(\gamma_j^+(t)\right)^2 e^{-\mu_j z} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_+(k,t)\, e^{ikz} dk.$$

$$q(y,t) = -2\frac{d}{dy} K(y,y,t).$$

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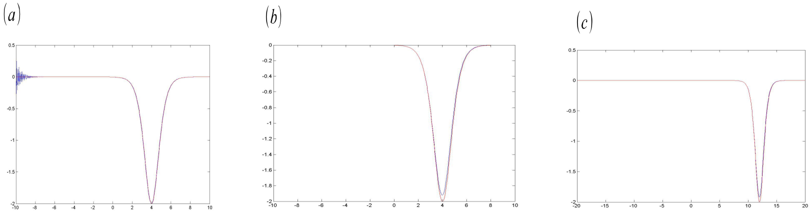
- Another version (**left** reflection coefficients):

$$L(y,r,t) + g(y+r,t) + \int_{-\infty}^{y} L(y,z,t)\, g(z+r,t)\, dz = 0, \;\; r < y$$

$$g(z,t) := \sum_{j=1}^{N} \left(\gamma_j^-(t)\right)^2 e^{\mu_j z} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_-(k,t)\, e^{-ikz} dk.$$

$$q(y,t) = 2\frac{d}{dy} L(y,y,t).$$

- Right reflection coeff. v.s. left reflection coeff.
- Example: one soliton solution of KdV equations.
  $f(z,t) = 2e^{-z+8t}$; $g(z,t) = 2e^{z-8t}$.
- (a): right reflection coefficients, $t = 1$, $(-10, 10)$, $h = 0.04$;
- (b) & (c): left reflection coefficients,
  (b): $t = 1$, $(-10, 10)$, $h = 0.04$;
  (c): $t = 3$, $(-20, 20)$, $h = 0.04$.
- Blue: IST solutions; red: one-soliton.



- Reference:
- O. Hald, *Numerical solution of the Gel'Fand-Levitan equation*, Linear Algebra Appl., 28 (1979), pp. 99–111.
- T. Aktosun, P.E. Sacks, *Potential splitting and numerical solution of the inverse scattering problem on the line*. Math. Methods Appl. Sci. 25 (2002), pp. 347–355.

- Consider IST w.r.t. the **left** reflection coeff.
- For $y \in [-n, n]$, divide $[-n, n]$ into N parts. Let $h = \frac{2n}{N}$,

$$\begin{aligned} y_i &= -n + (i-1)\,h, \\ r_j &= -n + (j-1)\,h, \quad , \; i, j, m = 1, ..., N+1. \\ z_m &= -n + (m-1)\,h, \end{aligned}$$

Let $L(y_i, r_j) = L_{ij}$, $g(y_i + r_j) = g_{ij}$,

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Let $L(y_i, r_j) = L_{ij}$, $g(y_i + r_j) = g_{ij}$,
then (GLM) can be approximated using the trapezoidal rule

$$L_{ij} + g_{ij} + h \sum_{m=1}^{i} \Delta_{im} L_{im} g_{jm} = 0, \quad 1 \le j \le i \qquad \text{(dGLM)}$$

where

$$\Delta_{ik} = \begin{cases} \frac{1}{2} & \text{for } k = 1, \; i, \\ 1 & \text{otherwise.} \end{cases}$$

- Recall that $q(y,t) = 2\frac{d}{dy}L(y,y,t)$.
- We need $L_{i,i}$ to get

$$q_i = q(y_i,t) = 2\left(\frac{L_{i+1,i+1} - L_{i,i}}{h}\right), i = 1, ..., N,$$

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- For $i = 1$, $j = 1$,
(i.e., $y = -\infty$ at
$L(y,r) + g(y+r) + \int_{-\infty}^{y} L(y,z)\, g(z+r)\, dz = 0$),
(dGLM) is degenerate.
In this case we set $L_{1,1} = -g_{1,1}$.

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  In this case we set $L_{1,1} = -g_{1,1}$.
- By the residue theorem,

$$g(z,t) = \begin{cases} \frac{q_0}{2} e^{\frac{-2q_0}{1-q_0^2}t + \frac{q_0}{2}z} & \text{for } z > 0, \\ \frac{q_0}{4} & \text{for } z = 0, \\ 0 & \text{for } z < 0. \end{cases}$$

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- $g_{ij} = g(y_i + r_j) = g(-2n + (i+j-2)h) = 0$ if
  $-2n + (i+j-2)h < 0$, i.e., $i + j < N + 2$.
  $g_{1,1} = 0$

- For general $i \geq 2$:
- Define

$$G := \begin{pmatrix} g_{1,1} & \cdots & g_{1,N+1} \\ \vdots & \ddots & \vdots \\ g_{N+1,1} & \cdots & g_{N+1,N+1} \end{pmatrix},$$

$G_i :=\text{prin}_{i+1}(G)$ be the principal sub-matrix of order $i+1$, and g_i^T be the last row of G_i.

- Similarly, define

$$L = \begin{pmatrix} L_{1,1} & \cdots & L_{1,N+1} \\ \vdots & \ddots & \vdots \\ L_{N+1,1} & \cdots & L_{N+1,N+1} \end{pmatrix},$$

$L_i = \text{prin}_{i+1}(L)$, and $\mathcal{L}_i^T$ be the last row of L_i.

- By writing $s_i = \text{diag}(\Delta_{i,1}, ..., \Delta_{i,i})$, (dGLM) can be expressed as

$$(I_{i+1} + hG_i s_i)\,\mathcal{L}_i = -g_i, \;\; i = 2, ..., N.$$

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$$L = \begin{pmatrix} L_{1,1} & \cdots & L_{1,N+1} \\ \vdots & \ddots & \vdots \\ L_{N+1,1} & \cdots & L_{N+1,N+1} \end{pmatrix},$$

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- By writing $s_i = \text{diag}(\Delta_{i,1}, ..., \Delta_{i,i})$, (dGLM) can be expressed as

$$(I_{i+1} + hG_i s_i)\,\mathcal{L}_i = -g_i, \;\; i = 2, ..., N.$$

- For larger i, the matrix size of $(I_{i+1} + hG_i s_i)\,\mathcal{L}_i = -g_i$ becomes larger also.

- Then consider IST to find $w(y,t)$ from $q(y,t)$.
- Constantin's version of IST is difficult (**nonlinear** processes).
- Constantin and Lenells's version (2003):
  Let $\phi(y,t)$ be the unique solution of $-\tilde{\psi}_{yy} + q(y,t)\tilde{\psi} = -\frac{1}{4}\tilde{\psi}$ with the asymptotic behavor
  $\phi(y,t) \approx e^{-\frac{y}{2}}$ and $\phi_y(y,t) \approx \frac{-1}{2}e^{-\frac{y}{2}}$ as $y \to \infty$.

  If $H_t : \mathbb{R} \to \mathbb{R}$ is the bijection given by

  $$H_t(y) = \int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi$$

  then

  $$m(x,t) = e^{2x}\phi^4\left(H_t^{-1}(e^x), t\right).$$
- (Note: $H_t(y) = e^x$.)

- Proof of $H_t(y) = \int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi = e^x$:
- Now $-\tilde{\psi}_{yy} + q(y,t)\tilde{\psi} = k^2\tilde{\psi}$ with $k^2 = -\frac{1}{4}$,
  recall $\lambda = \frac{1}{4} + k^2$ hence $\lambda = 0$,
  then (Lax1) becomes $-\psi_{xx} + \frac{1}{4}\psi = 0$ with solution $\psi(x,t) = e^{\frac{-x}{2}}$.
  recall $\tilde{\psi}(y,t) = m(x,t)^{\frac{1}{4}}\psi(x,t)$
  therefore $m(x,t)^{\frac{1}{4}} = \phi(y,t)\, e^{\frac{x}{2}}$, i.e.,

$$m(x,t) = e^{2x}\phi^4(y,t).$$

- Proof of $H_t(y) = \int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi = e^x$:
- Now $-\tilde{\psi}_{yy} + q(y,t)\,\tilde{\psi} = k^2 \tilde{\psi}$ with $k^2 = -\frac{1}{4}$,
  recall $\lambda = \frac{1}{4} + k^2$ hence $\lambda = 0$,
  then (Lax1) becomes $-\psi_{xx} + \frac{1}{4}\psi = 0$ with solution $\psi(x,t) = e^{\frac{-x}{2}}$.
  recall $\tilde{\psi}(y,t) = m(x,t)^{\frac{1}{4}}\,\psi(x,t)$
  therefore $m(x,t)^{\frac{1}{4}} = \phi(y,t)\,e^{\frac{x}{2}}$, i.e.,

$$m(x,t) = e^{2x}\phi^4(y,t).$$

- $\frac{dy}{dx} = m(x,t)^{\frac{1}{2}} = \phi^2(y,t)\,e^x$,

$$\frac{1}{\phi^2(y,t)} dy = e^x dx,$$

$$\int_{-\infty}^{y} \frac{1}{\phi^2(Y,t)} dY = \int_{-\infty}^{x} e^X dX$$

- $m(x,t)$ can then be expressed in a parametric form as follows with y being considered as a parameter.

$$\begin{cases} m(y,t) & = e^{2\ln H_t(y)}\phi^4(y,t), \\ & = H_t^2(y)\,\phi^4(y,t), \\ x(y,t) & = \ln H_t(y) = \ln \int_{-\infty}^{y} \frac{1}{\phi^4(\xi,t)}d\xi. \end{cases}$$

- We find the solution $\phi(y)$ of $-\phi_{yy} + q(y,t)\phi = -\frac{1}{4}\phi$ in Constantin & Lenelles by solving

$$\phi(y) = e^{-\frac{y}{2}} + \int_y^{\infty} \left(e^{\frac{\xi - y}{2}} - e^{\frac{y-\xi}{2}} \right) q(\xi)\,\phi(\xi)\,d\xi, \;\; y \in \mathbb{R}.$$

- Discrete version:

$$\phi(y_i) := \phi_i = e_i + h \sum_{k=i}^{N+1} \Delta^f_{i,k} E_{ik} q(\xi_k)\,\phi(\xi_k)\,,$$

where $E_{ik} = e^{\frac{\xi_k - y_i}{2}} - e^{\frac{y_i - \xi_k}{2}}$, $\xi_k = -n + (k-1)\,h$, $e_i = e^{-\frac{y_i}{2}}$,

$$\Delta^f_{ik} = \begin{cases} \frac{1}{2} & \text{for } k = i,\; k = N+1, \\ 1 & \text{otherwise.} \end{cases}$$

- That is,

$$\left(I_{N+1} - hE^0 \operatorname{diag}(q_i)\right)\Phi = \mathbf{e}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \vdots \\ \phi_{N+1} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ \vdots \\ \vdots \\ \vdots \\ e_{N+1} \end{pmatrix},$$

$$E^0 = \begin{pmatrix} 0 & E_{12} & \cdots & E_{1N} & \frac{E_{1,N+1}}{2} \\ 0 & 0 & \ddots & \vdots & \vdots \\ & \ddots & \ddots & E_{N-1,N} & \vdots \\ & \ddots & \ddots & 0 & \frac{E_{N,N+1}}{2} \\ 0 & & 0 & 0 & 0 \end{pmatrix}.$$

- Recall that $H_t(y) = \int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi = e^x$.
- Discrete version:

$$H_i := H(y_i) = h \sum_{k=1}^{i} \Delta_{i,k}^{H} \frac{1}{\phi^2(\xi_k)} \tag{dH}$$

where

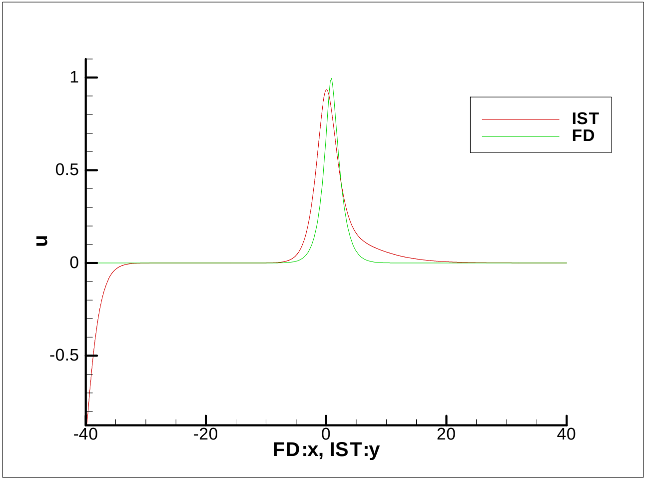
$$\Delta_{ik}^{H} = \begin{cases} \frac{1}{2} & \text{for } k = 1,\ k = i, \\ 1 & \text{otherwise.} \end{cases}$$

- For $i = 1$, we set $H_1 = 0$.
- Solve (dH) for $i = 2, ..., N+1$, leading to

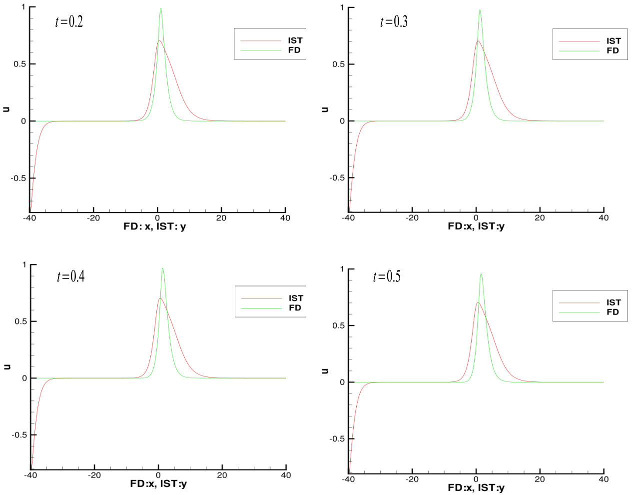
$$\begin{pmatrix} H_2 \\ \vdots \\ \vdots \\ H_{N+1} \end{pmatrix} = h \begin{pmatrix} \Delta^H_{2,1} & \Delta^H_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^H_{N+1,1} & \Delta^H_{N+1,2} & \cdots & \Delta^H_{N+1,N+1} \end{pmatrix} \begin{pmatrix} \frac{1}{\phi_1^2} \\ \vdots \\ \vdots \\ \frac{1}{\phi_{N+1}^2} \end{pmatrix}.$$

- Then $w_i = w(y_i, t) = H_i^2 \phi_i^4$, $i = 1, 2, ..., N+1$.
- Final step: recover u from m by solving $mu_{yy} + \frac{1}{2} m_y u_y - u = 1 - m$ through central difference approximation.

- Comparison between finite difference solutions and IST solutions: $(-40, 40)$, $h = 0.02$, $t = 0.1$.



- As time get larger, the IST solutions become more far away from the finite difference solutions.
  i.e., as time increasing, the error seems not easy to control.



Undergoing problems:

- Computational error from solving (dGLM).
- The errors from $\int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi$.
  In fact, we only compute $\int_{-n}^{y} \frac{1}{\phi^2(\xi,t)} d\xi$.

Undergoing problems:

- Computational error from solving (dGLM).
- The errors from $\int_{-\infty}^{y} \frac{1}{\phi^2(\xi,t)} d\xi$.
  In fact, we only compute $\int_{-n}^{y} \frac{1}{\phi^2(\xi,t)} d\xi$.
- Note: In $t = 0$, $\phi(y,0)$ can be found explicitly:

$$\phi(y,0) = \begin{cases} e^{\frac{-y}{2}}, & y \geq 0, \\ q_0 e^{\frac{y}{2}} + (1-q_0) e^{\frac{-y}{2}}, & y < 0. \end{cases}$$

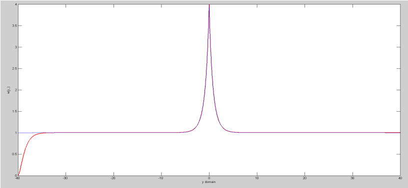
  Then also H_0 :

$$H_0(y) = \begin{cases} \frac{1}{1-q_0} \frac{1}{q_0+(1-q_0)e^{-y}}, & y \leq 0, \\ \frac{1}{1-q_0} + e^y - 1, & y > 0. \end{cases}$$

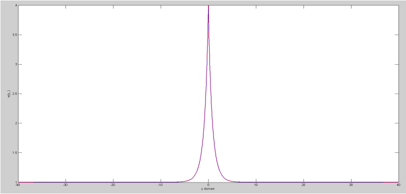
- Missing term is $H_0(-n) = \int_{-\infty}^{-n} \frac{1}{\phi^2(\xi,0)} d\xi = \frac{1}{1-q_0} \frac{1}{q_0+(1-q_0)e^n}$.

- $w(x,0)$ in $(-40,40)$ & $h = 0.08$ at $q_0 = 1/2$
  (a) without adding the missing term $H_0(-40)$;
  (b) add $H_0(-40)$.
  (Blue: exact $w(y,0)$; red: IST $w(y,0)$.)

(a)



(b)



Conclusion

- We construct a specific initial condition of Camassa-Holm equations such that the resulting scattering data have the non-zero reflection coefficients.
- Under the specific initial condition, the finite difference solution and the asymptotic solution and be compared at a large time. The oscillatory phenomena from the long-time asymptotics can be captured.
- About the numerical inverse scattering problem, at short time, the inverse scattering solution is not far away from the finite difference solution if the tails at minus infinity are neglected.

Thanks for Your Attention

and hope to get advice from you
to resolve the currently encountered problem.