# Semiclassical Initial-Boundary Value Problems for the Defocusing Nonlinear Schrödinger Equation

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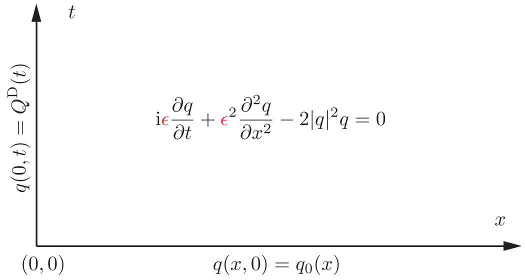
October 20, 2015
International Workshop on Integrable Systems
CASTS, National Taiwan University, Taipei

# The defocusing NLS equation on the half-line

## Dirichlet problem formulation. Well-posedness.




**Theorem (Carroll & Bu, *Appl. Anal.*, 41, 1991)**

*Suppose that* $q_0 \in H^2(\mathbb{R}_+)$ *and* $Q^{\mathrm{D}} \in C^2(\mathbb{R}_+)$*, and assume the compatibility condition* $q_0(0) = Q^{\mathrm{D}}(0)$*. Then (for every* $\epsilon > 0$*) there exists a unique global in time classical solution of the Dirichlet problem for the defocusing nonlinear Schrödinger equation on the half-line.*

Main question: how can the solution $q(x,t)$ be described in any detail?

Recall the Lax pair (Zakharov & Shabat, *Sov. Phys. JETP* **34**, 1972):

$$\epsilon\frac{\partial\Psi}{\partial x} = \begin{bmatrix} -\mathrm{i}k & q \\ q^* & \mathrm{i}k \end{bmatrix}\Psi$$

$$\epsilon\frac{\partial\Psi}{\partial t} = \begin{bmatrix} -2\mathrm{i}k^2 - \mathrm{i}|q|^2 & 2kq + \mathrm{i}\epsilon q_x \\ 2kq^* - \mathrm{i}\epsilon q_x^* & 2\mathrm{i}k^2 + \mathrm{i}|q|^2 \end{bmatrix}\Psi$$

The condition of simultaneous existence of a fundamental solution matrix $\Psi$ regardless of the value of the complex parameter $k$ is exactly that $q$ satisfy the defocusing nonlinear Schrödinger (NLS) equation:

$$\mathrm{i}\epsilon\frac{\partial q}{\partial t} + \epsilon^2\frac{\partial^2 q}{\partial x^2} - 2|q|^2 q = 0.$$

The Dirichlet problem can be transformed into a Riemann-Hilbert problem under some conditions[1]. First define spectral transforms: Let $Q^{\mathrm{N}}(t) := \epsilon q_x(0,t)$, and define special solutions of the Lax pair:

$$\epsilon \frac{d\mathbf{X}}{dx}(x;k) = \begin{bmatrix} -\mathrm{i}k & q_0(x) \\ q_0(x)^* & \mathrm{i}k \end{bmatrix} \mathbf{X}(x;k), \quad \lim_{x\to+\infty} \mathbf{X}(x;k)e^{\mathrm{i}kx\sigma_3/\epsilon} = \mathbb{I},$$

$$\epsilon \frac{d\mathbf{T}}{dt}(t;k) = \begin{bmatrix} -2\mathrm{i}k^2 - \mathrm{i}|Q^{\mathrm{D}}(t)|^2 & 2kQ^{\mathrm{D}}(t) + \mathrm{i}Q^{\mathrm{N}}(t) \\ 2kQ^{\mathrm{D}}(t)^* - \mathrm{i}Q^{\mathrm{N}}(t)^* & 2\mathrm{i}k^2 + \mathrm{i}|Q^{\mathrm{D}}(t)|^2 \end{bmatrix} \mathbf{T}(t;k),$$

$$\lim_{t\to+\infty} \mathbf{T}(t;k)e^{2\mathrm{i}k^2 t\sigma_3/\epsilon} = \mathbb{I}.$$

Then define a map $\{q_0, Q^{\mathrm{D}}, Q^{\mathrm{N}}\} \mapsto \{a, b, A, B\}$ by

$$a(k) := X_{22}(0;k),\ b(k) := X_{12}(0;k),\ A(k) := T_{22}(0;k),\ B(k) := T_{12}(0;k).$$

[1] A. S. Fokas, *A unified approach to boundary value problems*, SIAM, 2008.

# The defocusing NLS equation on the half-line

## Dirichlet problem. Integrable methodology.

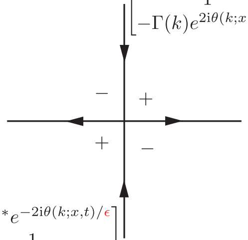
The spectral transforms $a$ and $b$ are analytic and bounded for $\Im\{k\} > 0$, while $A$ and $B$ are analytic and bounded for $\Im\{k^2\} > 0$. Now define

$$\gamma(k) := \frac{b(k)}{a(k)}, \quad \Gamma(k) := \frac{B(k^*)^*}{a(k)d(k)}, \quad \tilde{\gamma}(k) := \gamma(k) - \Gamma(k)^*,$$

where $d(k) := a(k)A(k^*)^* - b(k)B(k^*)^*$, and set $\theta(k;x,t) := kx + 2k^2t$. Then define a contour $\Sigma$ and a "jump matrix" $\mathbf{J}$ on $\Sigma \setminus \{0\}$ as:

$$\begin{bmatrix} 1 & 0 \\ -\Gamma(k)e^{2i\theta(k;x,t)/\epsilon} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\tilde{\gamma}(k)e^{-2i\theta(k;x,t)/\epsilon} \\ \tilde{\gamma}(k)^* e^{2i\theta(k;x,t)/\epsilon} & 1-|\tilde{\gamma}(k)|^2 \end{bmatrix}$$




$$\begin{bmatrix} 1-|\gamma(k)|^2 & \gamma(k)e^{-2i\theta(k;x,t)/\epsilon} \\ -\gamma(k)^* e^{2i\theta(k;x,t)/\epsilon} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \Gamma(k^*)^* e^{-2i\theta(k;x,t)/\epsilon} \\ 0 & 1 \end{bmatrix}$$

Then formulate a Riemann-Hilbert problem (RHP):

### Riemann-Hilbert Problem

*Seek* $\mathbf{M}(k;x,t)$, *a* $2\times 2$ *matrix function defined for* $k\in\mathbb{C}\setminus\Sigma$ *such that*

- $\mathbf{M}(\cdot;x,t)$ *is analytic in the four quadrants of its domain of definition.*
- *The boundary values* $\mathbf{M}_{\pm}(k;x,t)$ *taken by* $\mathbf{M}(k;x,t)$ *on* $\Sigma\setminus\{0\}$ *from* $\pm\Im\{k^2\}>0$ *are continuous and linked by the jump matrix:*

$$\mathbf{M}_{+}(k;x,t)=\mathbf{M}_{-}(k;x,t)\mathbf{J}(k;x,t),\quad k\in\Sigma\setminus\{0\}.$$

- $\mathbf{M}(k;x,t)\to\mathbb{I}$ *as* $k\to\infty$*.*

From the solution of this Riemann-Hilbert problem, define $q(x,t)$ by:

$$q(x,t):=2\mathrm{i}\lim_{k\to\infty}kM_{12}(k;x,t).$$

Then, $q(x,t)$ is a solution of the defocusing NLS equation.

# The defocusing NLS equation on the half-line

## Dirichlet problem. Integrable methodology.

The function $q(x,t)$ also satisfies $q(x,0) = q_0(x)$ and $q(0,t) = Q^{\mathrm{D}}(t)$ if:

- The given boundary data $\{Q^{\mathrm{D}}, Q^{\mathrm{N}}\}$ are consistent. That is, $Q^{\mathrm{N}}(t)$ agrees with ($\epsilon$ times) the Neumann boundary value of the solution of the (well-posed) Dirichlet problem with Dirichlet data $Q^{\mathrm{D}}$ and $q_0$.
- $d(k) \neq 0$ in the closed second quadrant of the complex $k$-plane. (Otherwise, poles must be admitted in $\mathbf{M}(k;x,t)$ with prescribed residue data.) J. Lenells recently posted a proof that $d(k) \neq 0$ for consistent boundary data $\{Q^{\mathrm{D}}, Q^{\mathrm{N}}\}$.

Problem: The spectral transforms $\{A, B\}$ cannot be calculated from the Dirichlet data $Q^{\mathrm{D}}$ alone; we also need to know the Neumann data $Q^{\mathrm{N}}$. Specifying both makes the problem overdetermined/inconsistent, so $q(x,t)$ (from the RHP) cannot generally satisfy the side-conditions.

A key role in the theory is therefore played by the *global relation*, an identity necessarily satisfied by the spectral transforms $\{a, b, A, B\}$ for consistent boundary data that encodes the Dirichlet-to-Neumann map in the spectral domain.

- In special situations (so-called *linearizable* boundary conditions) the global relation can be effectively solved by means of symmetries in the complex $k$-plane.
- Unfortunately, the only linearizable Dirichlet problem known corresponds to the homogeneous Dirichlet boundary condition

$$Q^{\mathrm{D}}(t) \equiv 0.$$

Of course this special case could be handled by the standard inverse-scattering transform on $\mathbb{R}$ by odd extension of $q_0$.

# The defocusing NLS equation on the half-line

## Iterative approach to the Dirichlet problem.

Since $q(x,t)$ from the RHP always satisfies defocusing NLS, consider (as an alternative to the global relation) an iterative scheme: given Dirichlet data $Q^{\mathrm{D}}(t)$ and $q_0(x)$, define $Q_0^{\mathrm{N}}(t)$ for $t>0$ as an *ad-hoc guess* for the unknown Neumann boundary data, and set $n=0$.

1. Set $Q^{\mathrm{N}}(t)=Q_n^{\mathrm{N}}(t)$, and together with $Q^{\mathrm{D}}(t)$ and $q_0(x)$ calculate the spectral transforms $\{a,b,A,B\}=\{a,b,A_n,B_n\}$.
2. Formulate the RHP with these spectral transforms and solve (unique solution off a "thin" exceptional set by analytic Fredholm theory). Obtain $q=q_n(x,t)$ solving defocusing NLS.
3. Define $Q_{n+1}^{\mathrm{N}}(t):=\epsilon\partial_x q_n(0,t)$ for $t>0$.
4. Set $n:=n+1$. Goto step 1.

We show that a modification of the first iteration of this scheme gives a good approximation to the solution of the boundary-value problem in the semiclassical limit $\epsilon\downarrow 0$.

How to get a good guess $Q_0^{\mathrm{N}}(t)$ for the Neumann data? Represent $q(x,t)$ in real phase-amplitude form:

$$q(x,t) = \eta(x,t)e^{\mathrm{i}\sigma(x,t)/\epsilon}, \quad \eta(x,t) := |q(x,t)|.$$

Then the defocusing NLS equation can be written exactly as a system:

$$\frac{\partial \eta}{\partial t} + 2\frac{\partial \sigma}{\partial x}\frac{\partial \eta}{\partial x} + \eta\frac{\partial^2 \sigma}{\partial x^2} = 0$$
$$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial \sigma}{\partial x}\right)^2 + 2\eta^2 = \frac{\epsilon^2}{\eta}\frac{\partial^2 \eta}{\partial x^2},$$

and the ratio of Neumann to Dirichlet boundary data takes the form:

$$-\mathrm{i}\frac{Q^{\mathrm{N}}(t)}{Q^{\mathrm{D}}(t)} = \frac{-\mathrm{i}\epsilon}{q(0,t)}\frac{\partial q}{\partial x}(0,t) = u(0,t) - \frac{\mathrm{i}\epsilon}{\eta(0,t)}\frac{\partial \eta}{\partial x}(0,t), \quad u(x,t) := \frac{\partial \sigma}{\partial x}(x,t).$$

Now consider the formal semiclassical limit $\epsilon \downarrow 0$:

$$-\mathrm{i}\frac{Q^{\mathrm{N}}(t)}{Q^{\mathrm{D}}(t)} = u(0,t) - \frac{\mathrm{i}\epsilon}{\eta(0,t)}\frac{\partial\eta}{\partial x}(0,t) \approx u(0,t).$$

But also (from defocusing NLS),

$$\frac{\partial\sigma}{\partial t}(0,t) + u(0,t)^2 + 2\eta(0,t)^2 = \frac{\epsilon^2}{\eta(0,t)}\frac{\partial^2\eta}{\partial x^2}(0,t) \approx 0.$$

For Dirichlet data of the form $Q^{\mathrm{D}}(t) := H(t)e^{\mathrm{i}S(t)/\epsilon}$, $H(t) := |Q^{\mathrm{D}}(t)|$,

$$-\mathrm{i}\frac{Q^{\mathrm{N}}(t)}{Q^{\mathrm{D}}(t)} \approx u(0,t) \quad \text{and} \quad S'(t) + u(0,t)^2 + 2H(t)^2 \approx 0.$$

Assuming that $S'(t) < -2H(t)^2$ for $t > 0$, eliminate $u(0,t)$ by

$$u(0,t) \approx U(t) := \sqrt{-S'(t) - 2H(t)^2} > 0.$$

The semiclassical approximation of the Dirichlet-to-Neumann map is:

$$Q^{\mathrm{N}}(t) \approx Q_0^{\mathrm{N}}(t) := \mathrm{i}U(t)Q^{\mathrm{D}}(t).$$

# The defocusing NLS equation on the half-line

## A modification of the first iteration.

For simplicity we consider zero initial data: $q_0(x) \equiv 0$.

We write $Q^{\mathrm{D}}(t) := H(t)e^{\mathrm{i}S(t)/\epsilon}$, where $S'(t) = -2H(t)^2 - U(t)^2$ and $H(\cdot)$ and $U(\cdot)$ are suitable given functions (more details soon...). Then

- $a(k) \equiv 1$ and $b(k) \equiv 0$ from the "$x$-problem" of the Lax pair.
- With $Q^{\mathrm{N}}(t)$ replaced by its formal approximation $\mathrm{i}U(t)Q^{\mathrm{D}}(t)$, the "$t$-problem" takes the form

$$\epsilon\frac{d\mathbf{T}}{dt}(t;k) = \begin{bmatrix} -2\mathrm{i}k^2 - \mathrm{i}H(t)^2 & (2k - U(t))H(t)e^{\mathrm{i}S(t)/\epsilon} \\ (2k - U(t))H(t)e^{-\mathrm{i}S(t)/\epsilon} & 2\mathrm{i}k^2 + \mathrm{i}H(t)^2 \end{bmatrix}\mathbf{T}(t;k).$$

This can be analyzed by WKB-type methods when $\epsilon > 0$ is small $\implies$ we can accurately and rigorously approximate $\{A_0(k), B_0(k)\}$.

Technical conditions on the functions $H : \mathbb{R}_+ \to \mathbb{R}$ and $U : \mathbb{R}_+ \to \mathbb{R}$:

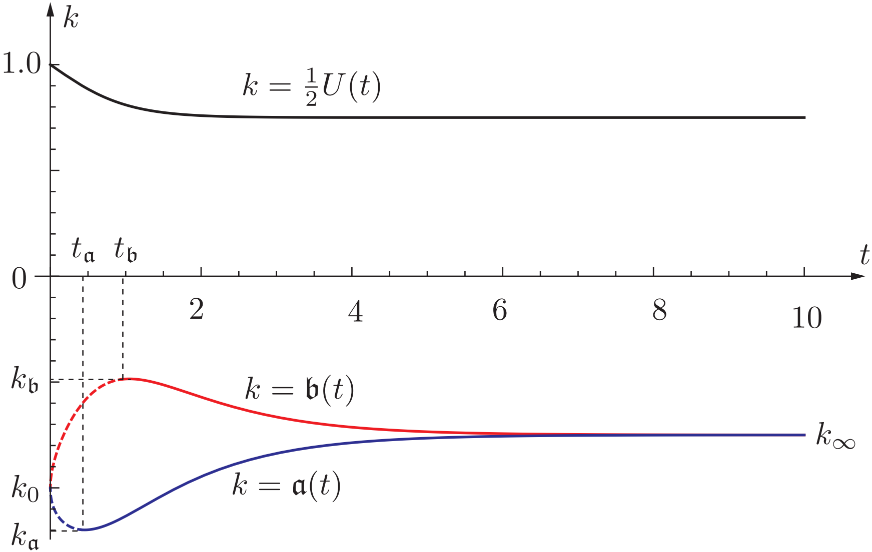
- $H(t)$ is real analytic and strictly positive for $t > 0$.
  - $H$ and all derivatives vanish as $t \to +\infty$ faster than any power of $t$.
  - There is some $h_0 > 0$ such that $H(t) = h_0 t^{1/2}(1 + o(1))$ and $H'(t) = \frac{1}{2} h_0 t^{-1/2}(1 + o(1))$ as $t \to 0$ with $\Re\{t\} \geq 0$.
- $U(t)$ is real analytic for $t > 0$ and $U(t) \geq 2H(t) + \delta$ for some $\delta > 0$.
  - $U'$ and all derivatives vanish as $t \to +\infty$ faster than any power of $t$.
  - There is a positive number $U_0$ such that $U(t) = U_0 + o(t^{1/2})$ and $U'(t) = \mathcal{O}(t^{-1/2})$ as $t \to 0$ with $\Re\{t\} \geq 0$.
- The functions

$$\mathfrak{a}(t) := -\frac{1}{2}U(t) - H(t) \quad \text{and} \quad \mathfrak{b}(t) := -\frac{1}{2}U(t) + H(t)$$

each have precisely one critical point in $(0, \infty)$, a non-degenerate maximum for $\mathfrak{b}$ and a non-degenerate minimum for $\mathfrak{a}$.

# The defocusing NLS equation on the half-line

## A modification of the first iteration.



Rigorous WKB analysis under these assumptions yields:

- Any zeros of the function $d_0(k) := A_0(k^*)^*$ in the second quadrant converge to $[k_\mathfrak{a}, k_\mathfrak{b}] \subset \mathbb{R}_-$ in the limit $\epsilon \downarrow 0$.
- $\Gamma_0(k) := B_0(k^*)^*/A_0(k^*)^* = \mathcal{O}(\epsilon^{1/2})$ uniformly for $k \in i\mathbb{R}$ and for $k < 0$ bounded away from $[k_\mathfrak{a}, k_\mathfrak{b}]$.
- Uniformly for $k$ in compact subsets of $(k_\mathfrak{a}, k_\mathfrak{b})$,

$$\Gamma_0(k) = \sqrt{1 - e^{-2\tau(k)/\epsilon}}e^{-2i\Phi(k)/\epsilon} + \mathcal{O}(\epsilon)$$
$$1 - |\Gamma_0(k)|^2 = e^{-2\tau(k)/\epsilon}(1 + \mathcal{O}(\epsilon)),$$

where with $s := \mathrm{sgn}(k^2 - k_0^2)$ and $t_-(k) < t_+(k)$ the roots of $(k - \mathfrak{a}(t))(k - \mathfrak{b}(t))$ (AKA "turning points"),

$$\Phi(k) := \frac{1}{2}S(0) + s\int_0^{t_-(k)} (U(t) - 2k))\sqrt{(k - \mathfrak{a}(t))(k - \mathfrak{b}(t))}\,dt$$
$$\tau(k) := \int_{t_-(k)}^{t_+(k)} (U(t) - 2k)\sqrt{(k - \mathfrak{a}(t))(\mathfrak{b}(t) - k)}\,dt.$$

Based on these asymptotics, we *replace* the jump matrices by their leading approximations, yielding a modified RHP. Let $\tilde{\Gamma}$ be defined on the real axis by:

$$\tilde{\Gamma}(k) := \chi_{(k_{\mathfrak{a}},k_{\mathfrak{b}})}(k)Y^{\epsilon}(k)e^{-2i\Phi(k)/\epsilon}, \quad Y^{\epsilon}(k) := \sqrt{1-e^{-2\tau(k)/\epsilon}}.$$

## Riemann-Hilbert Problem (modified first iteration)

*Seek* $\tilde{\mathbf{M}} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ *such that*

- $\tilde{\mathbf{M}}$ *is analytic taking boundary values* $\tilde{\mathbf{M}}_{\pm} : \mathbb{R} \to \mathbb{C}^{2\times 2}$ *from* $\mathbb{C}_{\pm}$.
- *The boundary values are related by the jump condition*

$$\tilde{\mathbf{M}}_{+}(k) = \tilde{\mathbf{M}}_{-}(k)\begin{bmatrix} 1-|\tilde{\Gamma}(k)|^2 & -\tilde{\Gamma}(k)^{*}e^{-2i\theta(k;x,t)/\epsilon} \\ \tilde{\Gamma}(k)e^{2i\theta(k;x,t)/\epsilon} & 1 \end{bmatrix}, \quad k \in \mathbb{R}.$$

- $\tilde{\mathbf{M}}(k) \to \mathbb{I}$ *as* $k \to \infty$.

Let

$$\tilde{q}^{\epsilon}(x,t) := 2\mathrm{i} \lim_{k\to\infty} k\tilde{M}_{12}(k).$$

It can be shown that $\tilde{q}^{\epsilon}(x,t)$ is for each $\epsilon > 0$ an infinitely differentiable solution of NLS. We prove the following additional results.

### Theorem (approximation of the initial condition)

*The solution* $q = \tilde{q}^{\epsilon}(x,t)$ *of the defocusing nonlinear Schrödinger equation satisfies*

$$\tilde{q}^{\epsilon}(x,0) = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2}), \quad x > 0, \quad \epsilon \to 0,$$

*where the error term is uniform on* $x \geq x_0$ *for each* $x_0 > 0$.

A similar result holds for certain nonzero $t$ as the following corollary (of the proof) shows…

Let $t \geq 0$, and let $X(t)$ denote the smallest nonnegative value of $x_0$ for which the inequality $x + 4kt - \Phi'(k) \geq 0$ holds for all $k \in (k_{\mathfrak{a}}, k_{\mathfrak{b}})$ whenever $x \geq x_0$.

### Corollary (existence of a vacuum domain)

*Let $t \geq 0$. The solution $q = \tilde{q}^{\epsilon}(x,t)$ satisfies $\tilde{q}^{\epsilon}(x,t) = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ as $\epsilon \downarrow 0$ whenever $x > X(t)$.*

Explicit asymptotes to $X(t)$ for small and large $t > 0$ are, respectively,

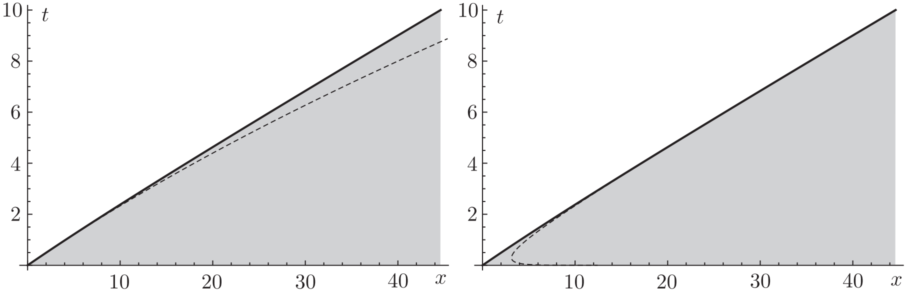
$$X_0(t) := -4k_0 t - \frac{h_0^2}{2k_0} t^2 \quad \text{and} \quad X_{\infty}(t) := -4k_{\mathfrak{a}} t - C_{\mathfrak{a}} \log(t),$$

where $C_{\mathfrak{a}}$ is a constant given by

$$C_{\mathfrak{a}} := \frac{1}{4} \frac{(U(t_{\mathfrak{a}}) - 2k_{\mathfrak{a}})\sqrt{\mathfrak{b}(t_{\mathfrak{a}}) - k_{\mathfrak{a}}}}{\sqrt{\frac{1}{2}\mathfrak{a}''(t_{\mathfrak{a}})}}.$$

# The defocusing NLS equation on the half-line

## A modification of the first iteration.




*The vacuum domain* $x > X(t)$ *(shaded) and the asymptotes* $x = X_0(t)$ *(left, dashed) and* $x = X_\infty(t)$ *(right, dashed) for* $H(t) := \frac{1}{2}t^{1/2}\mathrm{sech}(t)$ *and* $U(t) := 2 - \frac{1}{2}\tanh(t)$*.*

**Theorem (approximation of boundary conditions)**

*Suppose that $t > 0$ and $t \neq t_{\mathfrak{a}}$, $t \neq t_{\mathfrak{b}}$. The solution $q = \tilde{q}^{\epsilon}(x,t)$ of the defocusing nonlinear Schrödinger equation satisfies*

$$\tilde{q}^{\epsilon}(0,t) = H(t)e^{\mathrm{i}S(t)/\epsilon} + \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$$

$$\epsilon\tilde{q}^{\epsilon}_{x}(0,t) = \mathrm{i}U(t)H(t)e^{\mathrm{i}S(t)/\epsilon} + \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$$

*as $\epsilon \downarrow 0$, where the error terms are uniform for $t$ in compact subintervals of $(0,+\infty) \setminus \{t_{\mathfrak{a}}, t_{\mathfrak{b}}\}$.*

Again, the proof generalizes also for sufficiently small $x > 0$ as the following corollary shows…

## Corollary (existence of a plane-wave domain)

*Each point* $(0, t_0)$ *with* $t_0 > 0$ *and* $t_0 \neq t_{\mathfrak{a}}, t_{\mathfrak{b}}$ *has a neighborhood* $D_{t_0}$ *in the* $(x, t)$*-plane in which there exist unique differentiable functions* $\alpha(x, t)$ *and* $\beta(x, t)$ *satisfying* $\alpha(0, t) = \mathfrak{a}(t)$, $\beta(0, t) = \mathfrak{b}(t)$, *and*

$$\frac{\partial \alpha}{\partial t} - (3\alpha + \beta)\frac{\partial \alpha}{\partial x} = 0, \qquad \frac{\partial \beta}{\partial t} - (\alpha + 3\beta)\frac{\partial \beta}{\partial x} = 0.$$

*Moreover,* $\tilde{q}^{\epsilon}(x, t) = \eta(x, t)e^{i\sigma(x,t)/\epsilon} + \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ *holds uniformly for* $(x, t) \in D_{t_0}$ *as* $\epsilon \downarrow 0$, *where*

$$\eta(x, t) := \tfrac{1}{2}(\beta(x, t) - \alpha(x, t)) \quad \text{and} \quad \sigma(x, t) = S(t) - \int_0^x [\alpha(y, t) + \beta(y, t)]\, dy.$$

Note that $\eta(x, t)$ and $\sigma(x, t)$ satisfy the dispersionless defocusing NLS:

$$\frac{\partial \eta}{\partial t} + 2\frac{\partial \sigma}{\partial x}\frac{\partial \eta}{\partial x} + \eta\frac{\partial^2 \sigma}{\partial x^2} = 0, \qquad \frac{\partial \sigma}{\partial t} + \left(\frac{\partial \sigma}{\partial x}\right)^2 + 2\eta^2 = 0.$$

The second theorem and its corollary are proved by the Deift-Zhou steepest descent method for RHPs, specifically using genus zero $g$-function techniques. *[Another lecture. . . ]*

Proving the first theorem and its corollary involves showing that the RHP for $\tilde{\mathbf{M}}(k)$ can be transformed into a "small-norm problem," i.e., one for which the jump matrix is nearly $\mathbb{I}$. An algebraic factorization is required and technical obstructions arise due to:

- complicated behavior of the jump matrix factors near $k = k_{\mathfrak{a}}$ and $k = k_{\mathfrak{b}}$, and
- non-analyticity of the jump matrix factors at certain points in $(k_{\mathfrak{a}}, k_{\mathfrak{b}})$.

The latter analytical issue can be handled using the $\overline{\partial}$ steepest descent method, a generalization of the Deift-Zhou method.

The jump matrix (on $\mathbb{R}$) is exactly the identity except for $k_{\mathfrak{a}} < k < k_{\mathfrak{b}}$, where it admits the natural factorization

$$\begin{bmatrix} 1 - |\tilde{\Gamma}(k)|^2 & -\tilde{\Gamma}(k)^* e^{-2i\theta(k;x,t)/\epsilon} \\ \tilde{\Gamma}(k)e^{2i\theta(k;x,t)/\epsilon} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\tilde{\Gamma}(k)^* e^{-2i\theta(k;x,t)/\epsilon} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tilde{\Gamma}(k)e^{2i\theta(k;x,t)/\epsilon} & 1 \end{bmatrix}, \quad k_{\mathfrak{a}} < k < k_{\mathfrak{b}}.$$

Recall that $\tilde{\Gamma}(k) = Y^{\epsilon}(k)e^{-2i\Phi(k)/\epsilon}$ with $Y^{\epsilon}(k) = \sqrt{1 - e^{-2\tau(k)/\epsilon}} \approx 1$. If $\theta(k;x,t) - \Phi(k)$ is strictly increasing, we should try to deform the first/second factor into the lower/upper half-plane, “opening a lens” about the interval $[k_{\mathfrak{a}}, k_{\mathfrak{b}}]$.

However, we must proceed with care, because there are isolated points of non-analyticity of $\Phi$ and $\tau$, and hence of $\tilde{\Gamma}(k)$.

Assumptions in force on $U$ and $H$ $\Longrightarrow$

Important properties of $\tau : [k_{\mathfrak{a}}, k_{\mathfrak{b}}] \to \mathbb{R}$:

- $\tau(k)$ is analytic for $k \in [k_{\mathfrak{a}}, k_{\mathfrak{b}}] \setminus \{k_0, k_\infty\}$, and is $C^0$ near $k_0$ and $k_\infty$.
- $\tau(k) > 0$ holds strictly on $(k_{\mathfrak{a}}, k_{\mathfrak{b}})$.
- $\tau(k_{\mathfrak{a}}) = \tau(k_{\mathfrak{b}}) = 0$, while $\tau'(k_{\mathfrak{a}}) > 0$ and $\tau'(k_{\mathfrak{b}}) < 0$.

Important properties of $\Phi : [k_{\mathfrak{a}}, k_{\mathfrak{b}}] \to \mathbb{R}$:

- $\Phi(k)$ is analytic for $k \in (k_{\mathfrak{a}}, k_{\mathfrak{b}}) \setminus \{k_0\}$, and is $C^3$ near $k = k_0$.
- $\Phi'(k) \leq 0$ for $k_{\mathfrak{a}} < k < k_{\mathfrak{b}}$ with equality only for $k = k_0$.
- $\Phi$ has an analytic continuation $\Phi_{\mathfrak{a}}$ ($\Phi_{\mathfrak{b}}$) into the complex plane from a right (left) neighborhood of $k_{\mathfrak{a}}$ ($k_{\mathfrak{b}}$) satisfying

  $$\Phi_{\mathfrak{a},\mathfrak{b}}(k) = \Phi(k_{\mathfrak{a},\mathfrak{b}}) + C_{\mathfrak{a},\mathfrak{b}}(k - k_{\mathfrak{a},\mathfrak{b}})\log(|k - k_{\mathfrak{a},\mathfrak{b}}|) + \mathcal{O}(k - k_{\mathfrak{a},\mathfrak{b}}), \quad k \to k_{\mathfrak{a},\mathfrak{b}},$$

  where $C_{\mathfrak{a}} = \tau'(k_{\mathfrak{a}})/(2\pi) > 0$ and $C_{\mathfrak{b}} = -\tau'(k_{\mathfrak{b}})/(2\pi) > 0$.

In particular if the lens about $[k_{\mathfrak{a}}, k_{\mathfrak{b}}]$ is opened with nonzero acute angles at the endpoints $k_{\mathfrak{a}}$ and $k_{\mathfrak{b}}$, then

- $\tilde{\Gamma}(k) = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ holds uniformly near $k_{\mathfrak{a},\mathfrak{b}}$ along the lens boundary in $\mathbb{C}_+$, and
- $\tilde{\Gamma}(k^*)^* = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ holds uniformly near $k_{\mathfrak{a},\mathfrak{b}}$ along the lens boundary in $\mathbb{C}_-$.

The main idea behind this fact is that while the exponential decay of $e^{\mp 2i\Phi(k)/\epsilon}$ is not uniform near the endpoints, the factor

$$Y^{\epsilon}(k) = \sqrt{1 - e^{-2\tau(k)/\epsilon}}$$

vanishes like a square root. The net result is uniform (albeit very slow) decay as $\epsilon \downarrow 0$.

The factor $Y^\epsilon(k)$ fails to be analytic at $k_0, k_\infty$. But since $Y^\epsilon(k) - 1$ is exponentially small except near $k_\mathfrak{a}, k_\mathfrak{b}$, we can simply "leave it on $\mathbb{R}$" in the interior of $(k_\mathfrak{a}, k_\mathfrak{b})$ when we open the lens (details coming soon...).

The fact that $\Phi'(k) \leq 0$ on $(k_\mathfrak{a}, k_\mathfrak{b})$ suggests that we can use this monotonicity to obtain decay by deforming matrix factors into $\mathbb{C}_\pm$ (that's what steepest descent is all about). The non-analyticity of $\Phi$ at $k = k_0$ will be an obstruction.

We obtain an appropriate non-analytic extension of $\Phi(k)$ into the complex plane by following the $\overline{\partial}$ steepest descent method[2].

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[2] K. T.-R. McLaughlin and P. D. Miller, *Int. Math. Res. Not.* **2008**, 1–66, 2008.

Let $k_\mathrm{r} := \Re\{k\}$ and $k_\mathrm{i} := \Im\{k\}$. We first define a non-analytic extension of $\Phi(k_\mathrm{r})$ by the formula

$$\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i}) := \Phi(k_\mathrm{r}) + \mathrm{i}k_\mathrm{i}\Phi'(k_\mathrm{r}) + \frac{1}{2}(\mathrm{i}k_\mathrm{i})^2\Phi''(k_\mathrm{r}).$$

Note that $\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i})$ is *nearly analytic* near the real axis $k_\mathrm{i} = 0$ in the sense that

$$\overline{\partial}\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i}) = \frac{1}{2}\left(\frac{\partial}{\partial k_\mathrm{r}} + \mathrm{i}\frac{\partial}{\partial k_\mathrm{i}}\right)\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i}) = \frac{1}{4}(\mathrm{i}k_\mathrm{i})^2\Phi'''(k_\mathrm{r}) = \mathcal{O}(k_\mathrm{i}^2)$$

because $\Phi$ is three times continuously differentiable. Also, by Taylor’s formula,

$$\begin{aligned}
\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i}) - \Phi_\mathfrak{a}(k) &= \mathcal{O}(k_\mathrm{i}^3), \quad k_\mathfrak{a} < k_\mathrm{r} < k_0 \\
\hat{\Phi}_0(k_\mathrm{r}, k_\mathrm{i}) - \Phi_\mathfrak{b}(k) &= \mathcal{O}(k_\mathrm{i}^3), \quad k_0 < k_\mathrm{r} < k_\mathfrak{b}.
\end{aligned}$$

To get the $\mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ bound on $\tilde{\Gamma}$ near $k_{\mathfrak{a},\mathfrak{b}}$, we need the analytic functions $\Phi_{\mathfrak{a},\mathfrak{b}}$; but we are forced to use a non-analytic extension of $\Phi$ near $k_0$. Smoothly join them with a "bump function" $\mathcal{B} \in C^\infty(\mathbb{R};[0,1])$:

$$\text{for some small } \delta > 0, \quad \mathcal{B}(u) = \begin{cases} 1, & |u - k_0| < \delta \\ 0, & |u - k_0| > 2\delta. \end{cases}$$

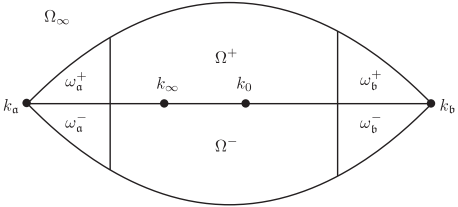
The extension of $\Phi$ that we will actually use is then given by the formula

$$\hat{\Phi}(k_r, k_i) := \begin{cases} \mathcal{B}(k_r)\hat{\Phi}_0(k_r, k_i) + (1 - \mathcal{B}(k_r))\Phi_{\mathfrak{a}}(k), & k_r \in (k_{\mathfrak{a}}, k_0] \\ \mathcal{B}(k_r)\hat{\Phi}_0(k_r, k_i) + (1 - \mathcal{B}(k_r))\Phi_{\mathfrak{b}}(k), & k_r \in [k_0, k_{\mathfrak{b}}). \end{cases}$$

Then $\overline{\partial}\hat{\Phi}(k_r, k_i) = \mathcal{O}(k_i^2)$ holds uniformly for $k_r \in (k_{\mathfrak{a}}, k_{\mathfrak{b}})$ because:

$$\overline{\partial}\hat{\Phi}(k_r, k_i) = \begin{cases} \mathcal{B}(k_r)\overline{\partial}\hat{\Phi}_0(k_r, k_i) + \overline{\partial}\mathcal{B}(k_r) \cdot (\hat{\Phi}_0(k_r, k_i) - \Phi_{\mathfrak{a}}(k)), & k_{\mathfrak{r}} \in (k_{\mathfrak{a}}, k_0] \\ \mathcal{B}(k_r)\overline{\partial}\hat{\Phi}_0(k_r, k_i) + \overline{\partial}\mathcal{B}(k_r) \cdot (\hat{\Phi}_0(k_r, k_i) - \Phi_{\mathfrak{b}}(k)), & k_r \in [k_0, k_{\mathfrak{b}}). \end{cases}$$

Now we open lenses. Consider these domains in the complex plane:

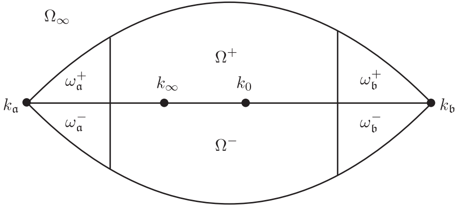



Make an explicit substitution $\tilde{\mathbf{M}}(k) \mapsto \mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}})$ by the following formulae: in the "bulk", we set

$$\mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}}) := \tilde{\mathbf{M}}(k)\begin{bmatrix} 1 & 0 \\ -e^{2\mathrm{i}(\theta(k;x,t)-\hat{\Phi}(k_{\mathrm{r}},k_{\mathrm{i}}))/\epsilon} & 1 \end{bmatrix}, \quad k \in \Omega^+,$$

$$\mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}}) := \tilde{\mathbf{M}}(k)\begin{bmatrix} 1 & -e^{2\mathrm{i}(\hat{\Phi}(k_{\mathrm{r}},k_{\mathrm{i}})-\theta(k;x,t))/\epsilon} \\ 0 & 1 \end{bmatrix}, \quad k \in \Omega^-,$$

Now we open lenses. Consider these domains in the complex plane:

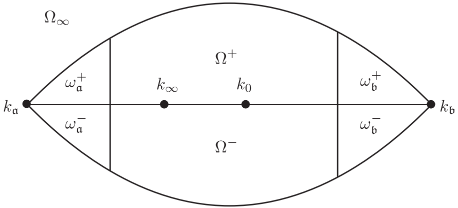



Make an explicit substitution $\tilde{\mathbf{M}}(k) \mapsto \mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}})$ by the following formulae: near $k_{\mathfrak{a},\mathfrak{b}}$, we set

$$\mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}}) := \tilde{\mathbf{M}}(k) \begin{bmatrix} 1 & 0 \\ -Y^{\epsilon}_{\mathfrak{a},\mathfrak{b}}(k) e^{2\mathrm{i}(\theta(k;x,t) - \Phi_{\mathfrak{a},\mathfrak{b}}(k))/\epsilon} & 1 \end{bmatrix}, \quad k \in \omega^+_{\mathfrak{a},\mathfrak{b}},$$

$$\mathbf{O}(k_{\mathrm{r}}, k_{\mathrm{i}}) := \tilde{\mathbf{M}}(k) \begin{bmatrix} 1 & -Y^{\epsilon}_{\mathfrak{a},\mathfrak{b}}(k) e^{2\mathrm{i}(\Phi_{\mathfrak{a},\mathfrak{b}}(k) - \theta(k;x,t))/\epsilon} \\ 0 & 1 \end{bmatrix}, \quad k \in \omega^-_{\mathfrak{a},\mathfrak{b}},$$

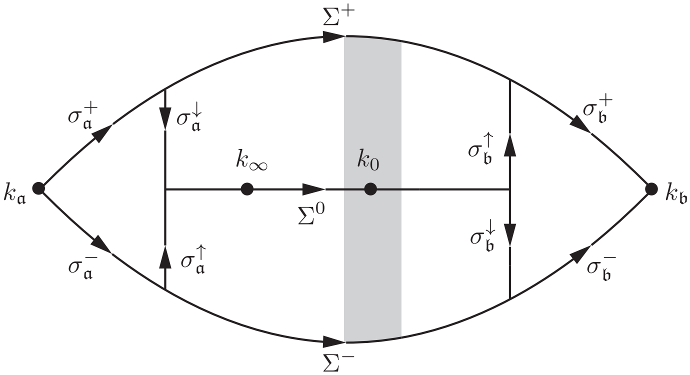
Now we open lenses. Consider these domains in the complex plane:




Make an explicit substitution $\tilde{\mathbf{M}}(k) \mapsto \mathbf{O}(k_\mathrm{r}, k_\mathrm{i})$ by the following formulae: and in the exterior domain, we set

$$\mathbf{O}(k_\mathrm{r}, k_\mathrm{i}) := \tilde{\mathbf{M}}(k), \quad k \in \Omega_\infty.$$

The matrix $\mathbf{O}$ has jump continuities across a contour $\Sigma$ illustrated here:




It is piecewise analytic except in the shaded region, a strip in the lens of width $4\delta$ centered at $k_\mathrm{r} = k_0$. $\mathbf{O}(k_\mathrm{r}, k_\mathrm{i})$ satisfies the conditions of a *hybrid Riemann-Hilbert-$\overline{\partial}$ problem*.

### Hybrid Riemann-Hilbert-$\overline{\partial}$ Problem

*Find a* $2 \times 2$ *matrix* $\mathbf{O}(k_\mathrm{r}, k_\mathrm{i})$ *with the following properties:*

- $\mathbf{O}$ *is continuous in each connected component of* $\mathbb{R}^2 \setminus \Sigma$ *taking continuous boundary values* $\mathbf{O}_\pm$ *on each oriented arc of* $\Sigma$.
- *On each oriented arc of* $\Sigma$ *there is a given and well-defined jump matrix* $\mathbf{J}_0(k_\mathrm{r}, k_\mathrm{i})$ *such that the boundary values* $\mathbf{O}_\pm$ *are related along that arc by the jump condition* $\mathbf{O}_+(k_\mathrm{r}, k_\mathrm{i}) = \mathbf{O}_-(k_\mathrm{r}, k_\mathrm{i})\mathbf{J}_0(k_\mathrm{r}, k_\mathrm{i})$.
- *On each connected component of* $\mathbb{R}^2 \setminus \Sigma$*, there is a given well-defined continuous matrix function* $\mathbf{W}$ *such that* $\overline{\partial}\mathbf{O}(k_\mathrm{r}, k_\mathrm{i}) = \mathbf{O}(k_\mathrm{r}, k_\mathrm{i})\mathbf{W}(k_\mathrm{r}, k_\mathrm{i})$ *holds.*
- $\mathbf{O}(k_\mathrm{r}, k_\mathrm{i}) \to \mathbb{I}$ *as* $(k_\mathrm{r}, k_\mathrm{i}) \to \infty$ *in* $\mathbb{R}^2$.

# The defocusing NLS equation on the half-line

## Proof of the “initial-data approximation” theorem.

When $t = 0$ and $x > 0$, this is a small-norm problem in the sense that:

- $\|\mathbf{J}_0 - \mathbb{I}\|_{L^\infty(\Sigma)} = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ and
- $\|\mathbf{W}\|_{L^\infty(\mathbb{R}^2\setminus\Sigma)} = \mathcal{O}(\epsilon)$.

These estimates depend on the following facts:

Fact #1: for $k_{\mathfrak{a}} < k < k_{\mathfrak{b}}$,

$$\theta'(k;x,0) - \Phi'(k) = x - \Phi'(k) \geq x > 0.$$

This is enough to control all of the analytic exponential factors (decay follows from the Cauchy-Riemann equations). It also controls the non-analytic exponential factors, since the exponents are dominated for small $k_{\mathrm{i}}$ (as the lens is sufficiently thin) by the linear terms in $\hat{\Phi}_0$, which again involve $\Phi'(k_{\mathrm{r}})$.

Fact #2: for $k$ bounded away from the endpoints $k_{\mathfrak{a}}$, $k_{\mathfrak{b}}$, $Y^{\epsilon}_{\mathfrak{a},\mathfrak{b}}(k) - 1$ is exponentially small as $\epsilon \downarrow 0$. This controls $\mathbf{J}_0 - \mathbb{I}$ on the real axis, where we've "left" $Y^{\epsilon}(k)$, and on the vertical contour segments $\sigma^{\uparrow,\downarrow}_{\mathfrak{a},\mathfrak{b}}$.

Facts #1 and #2 yield the estimate $\|\mathbf{J}_0 - \mathbb{I}\|_{L^\infty(\Sigma)} = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$.

Fact #3: from $\overline{\partial}\hat{\Phi}(k_{\mathrm{r}}, k_{\mathrm{i}}) = \mathcal{O}(k_{\mathrm{i}}^2)$, and the explicit formula

$$\mathbf{W}(k_{\mathrm{r}}, k_{\mathrm{i}}) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 2\mathrm{i}\epsilon^{-1}\overline{\partial}\hat{\Phi}(k_{\mathrm{r}}, k_{\mathrm{i}}) \cdot e^{2\mathrm{i}(\theta(k;x,t) - \hat{\Phi}(k_{\mathrm{r}},k_{\mathrm{i}}))/\epsilon} & 0 \end{bmatrix}, & k \in \Omega^+ \\ \begin{bmatrix} 0 & -2\mathrm{i}\epsilon^{-1}\overline{\partial}\hat{\Phi}(k_{\mathrm{r}}, k_{\mathrm{i}}) \cdot e^{2\mathrm{i}(\hat{\Phi}(k_{\mathrm{r}},k_{\mathrm{i}}) - \theta(k;x,t))/\epsilon} \\ 0 & 0 \end{bmatrix}, & k \in \Omega^-, \end{cases}$$

we get an estimate of the form $\|\mathbf{W}(k_{\mathrm{r}}, k_{\mathrm{i}})\| \le K\epsilon^{-1}k_{\mathrm{i}}^2 e^{-C|k_{\mathrm{i}}|/\epsilon}$ for $k \in \Omega^+ \cup \Omega^-$. Elsewhere, $\mathbf{W}$ vanishes identically. This yields the estimate $\|\mathbf{W}\|_{L^\infty(\mathbb{R}^2\setminus\Sigma)} = \mathcal{O}(\epsilon)$.

We use these estimates on $\mathbf{J}_0 - \mathbb{I}$ and $\mathbf{W}$ to solve the hybrid Riemann-Hilbert-$\overline{\partial}$ problem in two steps:

1. First, ignore the jump conditions altogether, and solve the "$\overline{\partial}$ part" of the problem.
2. Then use the solution of the "$\overline{\partial}$ part" as a parametrix and obtain a standard small-norm Riemann-Hilbert problem for the error.

The "$\overline{\partial}$ parametrix" solves the following problem.

### $\overline{\partial}$ Problem

*Find a $2 \times 2$ matrix $\dot{\mathbf{O}}(k_{\mathrm{r}}, k_{\mathrm{i}})$ with the following properties:*

- $\dot{\mathbf{O}} : \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ *is continuous.*
- $\overline{\partial}\dot{\mathbf{O}}(k_{\mathrm{r}}, k_{\mathrm{i}}) = \dot{\mathbf{O}}(k_{\mathrm{r}}, k_{\mathrm{i}})\mathbf{W}(k_{\mathrm{r}}, k_{\mathrm{i}})$ *holds in the distributional sense.*
- $\dot{\mathbf{O}}(k_{\mathrm{r}}, k_{\mathrm{i}}) \to \mathbb{I}$ *as* $(k_{\mathrm{r}}, k_{\mathrm{i}}) \to \infty$ *in* $\mathbb{R}^2$.

To solve the $\overline{\partial}$ problem, we set up an equivalent integral equation involving the solid Cauchy transform:

$$\dot{\mathbf{O}}(k_{\mathrm{r}},k_{\mathrm{i}})=\mathbb{I}+\mathcal{K}\dot{\mathbf{O}}(k_{\mathrm{r}},k_{\mathrm{i}})$$

where the action of the integral operator $\mathcal{K}$ is given by

$$\mathcal{K}\mathbf{F}(k_{\mathrm{r}},k_{\mathrm{i}}):=-\frac{1}{\pi}\iint_{\Omega^+\cup\Omega^-}\frac{\mathbf{F}(k_{\mathrm{r}}',k_{\mathrm{i}}')\mathbf{W}(k_{\mathrm{r}}',k_{\mathrm{i}}')\,dA(k_{\mathrm{r}}',k_{\mathrm{i}}')}{k'-k},\;dA(k_{\mathrm{r}},k_{\mathrm{i}}):=dk_{\mathrm{r}}\,dk_{\mathrm{i}}.$$

The operator norm of $\mathcal{K}$ acting on $L^\infty(\mathbb{R}^2)$ is easy to estimate because the Cauchy kernel is locally integrable on $\mathbb{R}^2$:

$$\|\mathcal{K}\|_{L^\infty(\mathbb{R}^2)\circlearrowleft}\le\frac{1}{\pi}\|\mathbf{W}\|_{L^\infty(\mathbb{R}^2)}\sup_{(k_{\mathrm{r}},k_{\mathrm{i}})\in\mathbb{R}^2}\iint_{\Omega^+\cup\Omega^-}\frac{dA(k_{\mathrm{r}}',k_{\mathrm{i}}')}{|k'-k|},$$

and the latter supremum is finite. Hence $\|\mathcal{K}\|_{L^\infty(\mathbb{R}^2)\circlearrowleft}=\mathcal{O}(\epsilon)$.

Iteration shows that $\dot{\mathbf{O}}(k_\mathrm{r}, k_\mathrm{i})$ is uniquely determined from the conditions of the $\overline{\partial}$ problem, and that $\|\dot{\mathbf{O}} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} = \mathcal{O}(\epsilon)$. In particular $\dot{\mathbf{O}}^{-1}$ exists for sufficiently small $\epsilon$ and $\|\dot{\mathbf{O}}^{-1} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} = \mathcal{O}(\epsilon)$.

Now use $\dot{\mathbf{O}}$ (solving the $\overline{\partial}$ problem) as a parametrix for $\mathbf{O}$ (solving the hybrid Riemann-Hilbert-$\overline{\partial}$ problem). Consider the substitution

$$\mathbf{E}(k_\mathrm{r}, k_\mathrm{i}) := \mathbf{O}(k_\mathrm{r}, k_\mathrm{i})\dot{\mathbf{O}}(k_\mathrm{r}, k_\mathrm{i})^{-1}.$$

As it is true for both factors, $\mathbf{E}(k_\mathrm{r}, k_\mathrm{i}) \to \mathbb{I}$ as $(k_\mathrm{r}, k_\mathrm{i}) \to \infty$. Also, by a direct calculation, one checks that for all $k \in \mathbb{C} \setminus \Sigma$, $\overline{\partial}\mathbf{E} = 0$. Therefore $\mathbf{E}$ is sectionally analytic and so we will write $\mathbf{E} = \mathbf{E}(k)$.

The jump of $\mathbf{E}(k)$ across the contour $\Sigma$ is easily obtained in terms of the "old" jump matrix $\mathbf{J}_0$ via conjugation by $\dot{\mathbf{O}}$:

$$\mathbf{E}_+(k) = \mathbf{E}_-(k)\dot{\mathbf{O}}(k_\mathrm{r}, k_\mathrm{i})\mathbf{J}_0(k_\mathrm{r}, k_\mathrm{i})\dot{\mathbf{O}}(k_\mathrm{r}, k_\mathrm{i})^{-1}, \quad k \in \Sigma.$$

Because

- $\dot{\mathbf{O}} = \mathbb{I} + \mathcal{O}(\epsilon)$ and $\dot{\mathbf{O}}^{-1} = \mathbb{I} + \mathcal{O}(\epsilon)$ uniformly on $\Sigma$, and
- $\mathbf{J}_0 = \mathbb{I} + \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ uniformly on $\Sigma$,

$\mathbf{E}_+(k) = \mathbf{E}_-(k)(\mathbb{I} + \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ holds uniformly on $\Sigma$. Therefore, for $\epsilon > 0$ sufficiently small, $\mathbf{E}$ satisfies the conditions of small-norm RHP in the $L^2(\Sigma)$ sense.
By standard arguments, $\mathbf{E}(k) - \mathbb{I} = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2})$ as $\epsilon \downarrow 0$ and

$$\mathbf{E}_1 := \lim_{k\to\infty} k(\mathbf{E}(k) - \mathbb{I}) = \mathcal{O}((\log(\epsilon^{-1}))^{-1/2}).$$

Unraveling the relationships $\tilde{\mathbf{M}} \to \mathbf{O} \to \mathbf{E}$ completes the proof. □

Proving the corollary amounts to the observation that the role of $x > 0$ and $t = 0$ in the proof was simply to provide the inequality (cf., Fact #1)

$$\theta'(k;x,0) - \Phi'(k) = x - \Phi'(k) \geq x > 0.$$

More generally, if $t \geq 0$, we can still have

$$\theta'(k;x,t) - \Phi'(k) = x + 4kt - \Phi'(k) > 0, \quad k \in (k_{\mathfrak{a}}, k_{\mathfrak{b}}),$$

provided that $x > 0$ is sufficiently large (given $t$). This condition defines the boundary $x = X(t)$ of the vacuum domain.

Note: if $f(\cdot) := -\Phi'(\cdot)$ is convex, then $X(t)$ may be explicitly given in terms of the Legendre dual $f^*$:

$$X(t) := f^*(-4t) = [-\Phi']^*(-4t), \quad t > 0, \quad f^*(p) := \sup_{k_{\mathfrak{a}}<k<k_{\mathfrak{b}}} (pk - f(k)).$$

# Conclusion

Semiclassical asymptotics and steepest descent techniques for Riemann-Hilbert and $\overline{\partial}$-problems can be combined with the so-called unified transform method ("inverse-scattering transform for initial-boundary-value problems") to produce accurate approximate solutions of non-homogeneous Dirichlet boundary-value problems for defocusing NLS without the use of the global relation.

Reference: P. D. Miller and Z.-Y. Qin, "Initial-boundary value problems for the defocusing nonlinear Schrödinger equation in the semiclassical limit," *Stud. Appl. Math.*, **134**, 276–362, 2015.

Thank You!