A focusing and defocusing complex short pulse equation

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Presentation at
International Workshop on Integrable Systems
Mathematical Analysis and Scientific Computing
National Taiwan University, Taipei

October 20, 2015
Outline of the talk

- Derivation of the complex short pulse (CSP) equation from nonlinear optics
- Bright, breather and rogue wave solutions to the focusing CSP equation.
- Dark soliton solution to the defocusing CSP equation
- Semi- and fully discrete analogues of the CSP equation
- Conclusion and further topics

Joint work with:
K. Maruno (Waseda University), Y. Ohta (Kobe University),
L. Ling (South China Univ. of Tech.), Z. Zhu (Shanghai Jiaotong Univ.)
Nonlinear Schrödinger (NLS) equation

\[ iq_t + q_{xx} + \sigma 2|q|^2 q = 0 , \quad \sigma = \pm 1 \]

- \( \sigma = 1 \): focusing case, possessing bright soliton
- \( \sigma = -1 \): defocusing case, possessing dark soliton
- Rogue wave solution for the focusing NLS equation
Review on the Nonlinear Schrödinger equation

Nonlinear Schrödinger (NLS) equation

\[ iq_t + q_{xx} + \sigma 2|q|^2q = 0, \quad \sigma = \pm 1 \]

- \( \sigma = 1 \): focusing case, possessing bright soliton
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- Rogue wave solution for the focusing NLS equation

Integrable semi-discrete NLS equation: Ablowitz-Ladik (AL) lattice

\[ i \frac{\partial q_n}{\partial t} + (1 + \sigma |q_n|^2)(q_{n-1} + q_{n+1}) = 0, \quad \sigma = \pm 1 \]
Coupled Nonlinear Schrödinger (CNLS) equation

\[ i q_{1,t} + q_{1,xx} + 2(|q_1|^2 + B|q_2|^2)q_1 = 0, \]

\[ i q_{2,t} + q_{2,xx} + 2(|q_2|^2 + B|q_1|^2)q_2 = 0. \]

- The parameter \( B \) is related to the ellipticity angle \( \theta \) as

\[ B = \frac{2 + 2 \sin^2 \theta}{2 + \cos^2 \theta}. \]

- For a linearly birefringent fiber \( (\theta = 0) \), \( B = \frac{2}{3} \), for a circularly birefringent fiber \( (\theta = \pi/2) \), \( B = 2 \). Only when \( B = 1 \), it is integrable (Manakov system)
Short pulse equation

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx} \]

- Sakovich & Sakovich (2005): A Lax pair of WKI type, linked to sine-Gordon equation through hodograph transformation;
- Matsuno (2007): Multisoliton solutions through Hirota’s bilinear method
Complex short pulse equation

\[
q_{xt} + q + \frac{1}{2}\sigma(|q|^2q_x)_x = 0, \quad (\sigma = \pm 1)
\]

- It is integrable which can be viewed as an analogue of the NLS equation in the ultra-short pulse region.
- It is more natural and appropriate in describing the propagation of the ultra-short pulses in compared with the short pulse equation
- \(\sigma = 1\): focusing case, bright soliton, breather and rogue wave solutions
- \(\sigma = -1\): defocusing case, dark soliton
Coupled complex short pulse equation

\begin{align*}
q_{1,xt} + q_1 + \frac{1}{2} \left((|q_1|^2 + B|q_2|^2)q_{1,x}\right)_x &= 0, \\
q_{2,xt} + q_2 + \frac{1}{2} \left((|q_2|^2 + B|q_1|^2)q_{2,x}\right)_x &= 0.
\end{align*}

- The parameter $B$ is related to the ellipticity angle $\theta$ same as the NLS equation.
- For a linearly birefringent fiber ($\theta = 0$), $B = \frac{2}{3}$, for a circularly birefringent fiber ($\theta = \pi/2$), $B = 2$.
- Similar to the Manakov system, only when $B = 1$, it is integrable.
Maxwell’s Equations:

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = -\frac{\partial D}{\partial t}.
\]

\[
D = \varepsilon E, \quad B = \mu H, \quad D = E + P.
\]

\(\varepsilon\): permittivity, \(\mu\): permeability. In vacuum, \(c^2 = 1/(\varepsilon_0\mu_0)\).
Maxwell’s Equations:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t}.
\]

\[
\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}.
\]

\(\varepsilon\): permittivity, \(\mu\): permeability. In vacuum, \(c^2 = 1/(\varepsilon_0 \mu_0)\).

In the frequency-dependent media,

\[
\mathbf{D} = \varepsilon * \mathbf{E}, \quad \mathbf{B} = \mu * \mathbf{H}.
\]

where \(\varepsilon = \varepsilon_0(1 + \chi^{(1)}(t))\). In the frequency domain

\[
\tilde{\mathbf{D}} = \tilde{\varepsilon}(\omega)\tilde{\mathbf{E}}, \quad \tilde{\mathbf{B}} = \tilde{\mu}(\omega)\tilde{\mathbf{H}}.
\]

\[
\nabla^2 \mathbf{E} - \frac{1}{c^2} \mathbf{E}_{tt} = \mu_0 \mathbf{P}_{tt},
\]

The induced polarization \(\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t)\).
Derivation of complex short pulse equation (II)

Assuming
\[ E = \frac{1}{2} e_1 (E(z, t) + c.c.) , \]
where \( E(z, t) \) is a complex-valued function.

\[ \tilde{E}_{zz}(z, \omega) + \tilde{\epsilon}(\omega) \frac{\omega^2}{c^2} \tilde{E}(z, \omega) = 0 , \]
where \( \tilde{E}(z, \omega) \) is the Fourier transform of \( E(z, t) \) defined as
\[ \tilde{E}(z, \omega) = \int_{-\infty}^{\infty} E(z, t) e^{i\omega t} dt , \]
and
\[ \tilde{\epsilon}(\omega) = 1 + \tilde{\chi}^{(1)}(\omega) . \]

where \( \tilde{\chi}^{(1)}(\omega) \) is the Fourier transform of \( \chi^{(1)}(t) \)
\[ \tilde{\chi}^{(1)}(\omega) = \int_{-\infty}^{\infty} \chi^{(1)}(t) e^{i\omega t} dt . \]
In the range ultra-short pulse, we approximate the response function $\chi^{(1)}(\lambda)$ by

$$\tilde{\chi}^{(1)}(\lambda) = \tilde{\chi}_0^{(1)} + \tilde{\chi}_2^{(1)} \lambda^2, \quad \tilde{\chi}_2^{(1)} > 0, \quad \lambda = \frac{2\pi c}{\omega}.$$ 

For Kerr media with cubic nonlinearity, $P_{NL}(z, t) = \epsilon_0 \epsilon_{NL} E(z, t)$

$$\epsilon_{NL} = \frac{3}{4} \chi^{(3)}_{xxxx} |E(z, t)|^2.$$ 

$$\ddot{E}_{zz} + \frac{1 + \tilde{\chi}_0^{(1)}}{c^2} \omega^2 \dot{E} + (2\pi)^2 \tilde{\chi}_2^{(1)} E + \epsilon_{NL} \frac{\omega^2}{c^2} \dot{E} = 0.$$
Applying the inverse Fourier transform yields a single nonlinear wave equation

\[ E_{zz} - \frac{1}{c_1^2} E_{tt} = \pm \frac{1}{c_2^2} E + \frac{3}{4} \chi^{(3)}_{xxxx} (|E|^2 E)_{tt} = 0. \]

Applying multiple scale expansion,

\[ E(z, t) = \epsilon E_0(\phi, z_1, z_2, \cdots) + \epsilon^2 E_1(\phi, z_1, z_2, \cdots) + \cdots, \]

where \( \epsilon \) is a small parameter, \( \phi \) and \( z_n \) are the scaled variables defined by

\[ \phi = \frac{t - \frac{x}{c_1}}{\epsilon}, \quad z_n = \epsilon^n z. \]

\[ - \frac{2}{c_1} \frac{\partial^2 E_0}{\partial \phi \partial z_1} = \pm \frac{1}{c_2^2} E_0 + \frac{3}{4} \chi^{(3)}_{xxxx} \frac{\partial}{\partial \phi} \left( |E_0|^2 \frac{\partial E_0}{\partial \phi} \right). \]
By a scale transformation

\[ x = \frac{c_1}{2} \phi, \quad t = c_2 z_1, \quad q = \frac{c_1 \sqrt{6 c_2 \chi^{(3)}_{xxxx}}}{4} E_0 \]

we have

\[ q_{xt} \pm q + \frac{1}{2} (|q|^2 q_x)_x = 0 \]

\[ q_{xt} + q + \frac{1}{2} \sigma (|q|^2 q_x)_x = 0, \quad \sigma = \pm 1. \]
Focusing and defocusing complex short pulse equation

By a scale transformation

\[ x = \frac{c_1}{2} \phi, \quad t = c_2 z, \quad q = \frac{c_1 \sqrt{6 c_2 \chi_x^{(3)}}}{4} E_0 \]

we have

\[ q_{xt} \pm q + \frac{1}{2} \left( |q|^2 q_x \right)_x = 0 \]

\[ q_{xt} + q + \frac{1}{2} \sigma \left( |q|^2 q_x \right)_x = 0, \quad \sigma = \pm 1. \]

Coupled complex short pulse equation of mixed type

\[
\begin{aligned}
q_{1,xt} + q_1 + \frac{1}{2} \left( (\sigma_1 |q_1|^2 + \sigma_2 |q_2|^2) q_{1,x} \right)_x &= 0, \\
q_{2,xt} + q_2 + \frac{1}{2} \left( (\sigma_1 |q_1|^2 + \sigma_2 |q_2|^2) q_{2,x} \right)_x &= 0
\end{aligned}
\]

- focusing-focusing \((\sigma_1 = \sigma_2 = 1)\); defocusing-defocusing \((\sigma_1 = \sigma_2 = -1)\) and focusing-defocusing \((\sigma_1 = 1; \sigma_2 = -1)\).

- Bright, dark and bright-dark soliton solutions and rogue wave solution
Complex coupled dispersionless (CCD) equation

\[
\begin{cases}
q_{ys} = \rho q, \\
\rho_s \pm \frac{1}{2}(|q|^2)_y = 0
\end{cases}
\]

- Only the positive sign was studied
From the complex coupled dispersiveless equation to the complex short pulse equation

\[
\begin{cases}
q_{ys} = \rho q, \\
\rho_s \pm \frac{1}{2} (|q|^2)_y = 0
\end{cases}
\]

We define a hodograph transformation

\[dx = \rho dy \mp \frac{1}{2} |q|^2 ds, \quad dt = -ds,\]

then we have

\[\partial_y = \rho^{-1} \partial_x, \quad \partial_s = -\partial_t \mp \frac{1}{2} |q|^2 \partial_x\]

Accordingly, the equation \( q_{ys} = \rho q \) gives the

\[\partial_x (-\partial_t \mp \frac{1}{2} |q|^2 \partial_x) q = q,\]

\[q_{xt} + q \pm \frac{1}{2} (|q|^2 q_x)_x = 0.\]
Bilinear equations of the focusing complex short pulse equation

**Theorem**

The focusing complex short pulse equation

\[ q_{xt} + q + \frac{1}{2} (|q|^2 q_x)_x = 0 \]

can be derived from bilinear equations

\[ D_s D_y f \cdot g = fg, \quad D_s^2 f \cdot f = \frac{1}{2} |g|^2, \]

through the hodograph transformation

\[ x = y - 2(\ln f)_s, \quad t = -s \]

and the dependent variable transformation \[ q = \frac{g}{f} \]
Theorem

The CSP equation admits multi-soliton solution

\[ f = \begin{bmatrix} A & I \\ -I & B \end{bmatrix}_{2N \times 2N}, \quad g = \begin{bmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_1 & 0 \end{bmatrix}_{(2N+1) \times (2N+1)}, \]

where the elements defined respectively by

\[ a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{-1})} e^{\xi_i + \xi_j^*}, \quad b_{ij} = \frac{\alpha_i \alpha_j^*}{2(p_j^{-1} + p_i^{-1})} \]

\[ \xi_i = p_i y + \frac{1}{p_i} s + \xi_i^0, \quad \xi_j^* = p_j^* y + \frac{1}{p_j^*} s + \xi_j^{*0}, \]
One-soliton to the focusing complex SP equation

\[ f = 1 + \frac{1}{4} \frac{|\alpha_1|^2 (p_1 \bar{p}_1)^2}{(p_1 + \bar{p}_1)^2} e^{\eta_1 + \bar{\eta}_1}, \quad g = \alpha_1 e^{\eta_1}. \]

Let \( p_1 = p_1 R + i p_1 I \)

\[ q = \frac{\alpha_1}{|\alpha_1|} \frac{2p_1 R}{|p_1|^2} e^{i\eta_1 I} \text{sech} (\eta_1 R + \eta_{10}), \]

\[ x = y - \frac{2p_1 R}{|p_1|^2} (\tanh (\eta_1 R + \eta_{10}) + 1), \quad t = -s, \]

When \( p_1 R < p_1 I \), the solution is a smooth envelop soliton; when \( p_1 R = p_1 I \), the solution becomes a cuspon soliton.

Figure: Illustration of smooth and cuspon soliton for focusing CSP equation
Define the following tau-functions for two-component KP hierarchy,

\[ f_{mn} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \]

\[ g_{mn} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & -\bar{\Phi} & 0 \end{vmatrix}, \quad h_{mn} = \begin{vmatrix} A & I & 0^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & 0 & 0 \end{vmatrix}, \]

where \( A \) and \( B \) are \( N \times N \) matrices whose elements are

\[ a_{ij} = \frac{1}{p_i + \bar{p}_j} \left( \frac{p_i}{\bar{p}_j} \right)^n e^{\xi_i + \bar{\xi}_j}, \quad b_{ij} = \frac{1}{q_i + \bar{q}_j} \left( \frac{q_i}{\bar{q}_j} \right)^m e^{\eta_i + \bar{\eta}_j}, \]

with

\[ \xi_i = \frac{1}{p_i} x_{-1} + p_i x_1 + \xi_{i0}, \quad \bar{\xi}_j = \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \bar{\xi}_{j0}, \]

\[ \eta_i = q_i y_1 + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1 + \bar{\eta}_{j0}, \]
Two-component KP hierarchy and its Gram-type solution

\( \Phi, \Psi, \bar{\Phi} \) and \( \bar{\Psi} \) are \( N \)-component row vectors

\[
\Phi = (p_1^ne^{\xi_1}, \ldots, p_N^ne^{\xi_N}), \quad \bar{\Phi} = \left((-\bar{p}_1)^{-n}e^{\xi_1}, \ldots, (-\bar{p}_N)^{-n}e^{\xi_N}\right),
\]

\[
\Psi = (q_1^me^{\eta_1}, \ldots, q_N^me^{\eta_N}), \quad \bar{\Psi} = \left((-\bar{q}_1)^{-m}e^{\eta_1}, \ldots, (-\bar{q}_N)^{-m}e^{\eta_N}\right).
\]

Then the following bilinear equations hold

\[
\frac{1}{2}D_x D_y f_{nm} \cdot f_{nm} = -g_{nm}h_{nm},
\]

\[
D_{x_{-1}}g_{nm} \cdot f_{nm} = g_{n-1,m}f_{n+1,m},
\]

\[
(D_x D_{x_{-1}} - 2)g_{nm} \cdot f_{nm} = -D_x g_{n-1,m} \cdot f_{n+1,m},
\]

\[
D_x g_{n,m+1} \cdot f_{n+1,m} = g_{n+1,m+1}f_{nm}.
\]
Recall the bilinear equation of the CSP equation

\[ D_s D_y f \cdot g = f g, \quad D_s^2 f \cdot f = \frac{1}{2} |g|^2, \]

Task: How to get them from the following bilinear equations of two-component KP?

\[ \frac{1}{2} D_{x_1} D_{y_1} f_{nm} \cdot f_{nm} = -g_{nm} h_{nm}, \]

\[ D_{x_{-1}} g_{nm} \cdot f_{nm} = g_{n-1,m} f_{n+1,m}, \]

\[ (D_{x_1} D_{x_{-1}} - 2) g_{nm} \cdot f_{nm} = -D_{x_1} g_{n-1,m} \cdot f_{n+1,m}, \]

\[ D_{x_1} g_{n,m+1} \cdot f_{n+1,m} = g_{n+1,m+1} f_{nm}. \]
Reductions to the CSP equation (II)

Under the condition $q_j = \bar{p}_j$, $\bar{q}_j = p_j$ we have

$$f_{n+1,m+1} = f_{nm}, \quad g_{n+1,m+1} = -g_{nm},$$

$$\partial x_1 f_{nm} = \partial y_1 f_{nm}, \quad \partial x_1 g_{nm} = \partial y_1 g_{nm}.$$ 

it then follows

$$(D_{x_1} D_{x_{-1}} - 2) g_{nm} \cdot f_{nm} = D_{x_1} g_{n,m+1} \cdot f_{n+1,m}$$

$$= g_{n+1,m+1} f_{nm}$$

$$= -g_{nm} f_{nm}$$

from

$$(D_{x_1} D_{x_{-1}} - 2) g_{nm} \cdot f_{nm} = -D_{x_1} g_{n-1,m} \cdot f_{n+1,m},$$

$$D_{x_1} g_{n,m+1} \cdot f_{n+1,m} = g_{n+1,m+1} f_{nm}.$$
\[ \partial x_1 f_{nm} = \partial y_1 f_{nm}, \quad \partial x_1 g_{nm} = \partial y_1 g_{nm}. \]

From
\[ \frac{1}{2} D_{x_1} D_{y_1} f_{nm} \cdot f_{nm} = -g_{nm} h_{nm}, \]

it then follows
\[ \frac{1}{2} D_{x_1}^2 f_{nm} \cdot f_{nm} = -g_{nm} h_{nm}. \]

Let \( f = f_{00}, \ g = g_{00}, \ h = h_{00}, \) the above bilinear equations read
\[ (D_{x_1} D_{x_{-1}} - 1) g \cdot f = 0, \]
\[ \frac{1}{2} D_{x_1}^2 f \cdot f = -g h. \]
By taking
\[ \bar{p}_j = p_j^* , \quad \bar{\xi}_{j0} = \xi_{j0}^* , \quad \bar{\eta}_{j0} = \eta_{j0}^* , \]
we can easily check that \( f \) is real and \( h = -g^* \). Then
\[
(D_{x_1} D_{x_{-1}} - 1) g \cdot f = 0 ,
\]
\[
D_{x_1}^2 f \cdot f = 2|g|^2 .
\]
By taking
\[ \tilde{p}_j = p_j^*, \quad \tilde{\xi}_{j0} = \xi_{j0}^*, \quad \tilde{\eta}_{j0} = \eta_{j0}^*, \]
we can easily check that \( f \) is real and \( h = -g^* \). Then
\[ (D_{x_1} D_{x_{-1}} - 1)g \cdot f = 0, \]
\[ D_{x_1}^2 f \cdot f = 2|g|^2. \]

By variable transformation
\[ s = 2(x_1 + y_1), \quad y = \frac{1}{2}(x_{-1} + y_{-1}), \]
we arrive at the bilinear equations for the CSP equation. The multi-soliton solution can be obtained by a reparametrization
\[ p_i \rightarrow 2p_i^{-1}, \quad p_i^* \rightarrow 2p_i^{*-1}, \]
It is known that the CCD equation admits the following Lax pair

$$\Psi_y = U(\rho, q; \lambda)\Psi, \quad \Psi_s = V(q; \lambda)\Psi,$$

where

$$U(\rho, q; \lambda) = \begin{bmatrix} -\frac{i\rho}{\lambda} & -\frac{q^*_y}{\lambda} \\ \frac{q_y}{\lambda} & \frac{i\rho}{\lambda} \end{bmatrix}, \quad V(q; \lambda) = \begin{bmatrix} \frac{i}{4}\lambda & \frac{iq^*_x}{2} \\ \frac{iq}{2} & -\frac{i}{4}\lambda \end{bmatrix}$$

Through the reciprocal transformation:

$$dx = \rho dy - \frac{1}{2}|q|^2 ds, \quad dt = -ds,$$

one can obtain the CSP equation and its Lax pair:

$$\Psi_x = \begin{bmatrix} -\frac{i}{\lambda} & -\frac{q^*_x}{\lambda} \\ \frac{q_x}{\lambda} & \frac{i}{\lambda} \end{bmatrix} \Psi,$$

$$\Psi_t = \begin{bmatrix} -\frac{i}{4}\lambda + \frac{i|q|^2}{2\lambda} & -\frac{iq^*_x}{2} + \frac{|q|^2 q^*_x}{2\lambda} \\ -\frac{iq}{2} - \frac{|q|^2 q_x}{2\lambda} & \frac{i}{4}\lambda - \frac{i|q|^2}{2\lambda} \end{bmatrix} \Psi.$$
The Darboux transformation for the focusing CSP equation

Theorem

The Darboux matrix

\[ T = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*} P_1, \quad P_1 = \frac{|y_1\rangle\langle y_1|}{\langle y_1|y_1\rangle}, \quad |y_1\rangle = \begin{bmatrix} \psi_1(x, t; \lambda_1) \\ \phi_1(x, t; \lambda_1) \end{bmatrix} \]

can convert the Lax pair of the CSP eq. \( \Psi_y = U(q; \lambda) \Psi, \Psi_s = V(q; \lambda) \Psi \)
into a new system

\[ \Psi[1]_y = U(q; \lambda) \Psi[1], \quad \Psi[1]_s = V(q; \lambda) \Psi[1]. \]

The Bäcklund transformations between \((q[1], \rho[1])\) and \((q, \rho)\) are given through

\[ \rho[1] = \rho - 2 \ln_{y_s} \left( \frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1} \right), \]

\[ q[1] = q + \frac{\lambda_1^* - \lambda_1}{\langle y_1|y_1\rangle} \psi_1^* \phi_1. \]
Single breather solution

We start with a seed solution

\[ \rho[0] = -\frac{\gamma}{2}, \quad q[0] = \frac{\beta}{2}e^{i\theta}, \quad \theta = y + \frac{\gamma}{2}s. \]

Then we can get the single breather solution

\[ q[1] = \frac{\beta}{2} \left[ \frac{\cosh 2(\theta_{1,R} - i\varphi_{1,I}) \cosh(\varphi_{1,R}) + \sin 2(\theta_{1,I} + i\varphi_{1,R}) \sin(\varphi_{1,I})}{\cosh(2\theta_{1,R}) \cosh(\varphi_{1,R}) - \sin(2\theta_{1,I}) \sin(\varphi_{1,I})} \right] \]

\[ x = -\gamma y - \frac{\beta^2}{8} s - 2 \ln s \left[ \cosh(2\theta_{1,R}) \cosh(\varphi_{1,R}) - \sin(2\theta_{1,I}) \sin(\varphi_{1,I}) \right], \]

\[ t = -s, \]
Multi-breather solution to the CSP equation

Generally, $N$-breather solution:

$$q[N] = \frac{\beta}{2} \left[ \frac{\text{det}(G)}{\text{det}(M)} \right] e^{i\theta},$$

$$x = -\frac{\gamma}{2} y - \frac{\beta^2}{8} s - 2 \ln s (\text{det}(M)), \quad t = -s,$$

where

$$M = \left( \left[ \frac{e^{2(\theta_i^* + \theta_j)}}{\xi_i^* - \xi_j} + \frac{e^{2\theta_i^*}}{\xi_i^* - \chi_j} + \frac{e^{2\theta_j}}{\chi_i^* - \xi_j} + \frac{1}{\chi_i^* - \chi_j} \right] e^{-(\theta_i^* + \theta_j)} \right)_{1 \leq i, j \leq N},$$

$$G = \left( \left[ \frac{\xi_i^* + \gamma e^{2(\theta_i^* + \theta_j)}}{\xi_j + \gamma \xi_i^* - \xi_j} + \frac{\xi_i^* + \gamma e^{2\theta_i^*}}{\chi_j + \gamma \xi_i^* - \chi_j} + \frac{\chi_i^* + \gamma e^{2\theta_j}}{\xi_j + \gamma \chi_i^* - \xi_j} + \frac{\chi_i^* + \gamma}{\chi_j + \gamma \chi_i^* - \chi_j} \right] e^{-(\theta_i^* + \theta_j)} \right)_{1 \leq i, j \leq N}.$$
Rogue wave solution to the CSP equation

\[ q[1] = \frac{\beta}{2} \left[ 1 + \frac{16(i\beta^2 y - \beta^2 - \gamma^2)}{\beta^2 (2y - \gamma s)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2} \right] e^{i\theta}, \]

\[ x = -\frac{\gamma}{2} y - \frac{\beta^2}{8} s - \frac{4\beta^2 (\gamma^2 s + \beta^2 s - 2\gamma y)}{\beta^2 (2y - \gamma s)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2}, \quad t = -s. \]

- \( \beta^2 < \frac{\gamma^2}{3} \), then we can obtain the regular rogue wave solution
- \( \beta^2 = \frac{\gamma^2}{3} \), then we can obtain the cuspon-type rogue wave
- \( \beta^2 > \frac{\gamma^2}{3} \), then we can obtain the loop-type rogue wave solution
Rogue wave solution to the focusing complex SP equation

First and second-order rogue wave solutions

Figure: Illustration for the 1st and 2nd rogue waves of the focusing CSP equation
Bilinear equations of the defocusing complex short pulse equation

The complex short pulse equation

\[ q_{xt} + q - \frac{1}{2} \left( |q|^2 q_x \right)_x = 0 \]

can be derived from bilinear equations

\[ (D_s D_y - i\omega D_y + i\kappa D_s)g \cdot f = 0, \quad D_s^2 f \cdot f = \frac{1}{2}\omega^2 \left( f^2 - |g|^2 \right), \]

through the hodograph transformation

\[ x = \omega \kappa y + \frac{\omega}{2} s - 2(\ln f)_s, \quad t = -s \]

and the dependent variable transformation \( q = \frac{g}{f} e^{i(\kappa y - \omega s)} \)
Multi dark soliton to the defocusing complex short pulse equation

\[ f = |A|, \quad g = |A'|, \]

where the elements defined respectively by

\[ a_{ij} = \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*}, \quad a_{ij}' = \delta_{ij} + \left( -\frac{p_i}{p_j^*} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \]

\[ \xi_i = \frac{\omega}{2} p_i s + q_i \kappa y + \xi_{i0}, \quad \xi_i^* = \frac{\omega}{2} p_i^* s + q_i^* \kappa y + \xi_{i0}^* \]

\[ q_i = \frac{1}{p_i - i}, \quad q_i^* = \frac{1}{p_i^* + i} \]

where \( |p_i| = 1 = e^{i\phi}, \quad p_i^* = e^{-i\phi} \).
Define the following tau-functions for the single KP hierarchy with negative flow:

\[ \tau_{nk} = \left| m_{ij}^{nk} \right|_{1 \leq i, j \leq N} = \left| \delta_{ij} + \frac{1}{p_i + \bar{p}_j} \varphi_i^{nk} \psi_j^{nk} \right| \]

where

\[ \varphi_i^{nk} = p_i^n (p_i - a)^k e^{\xi_i} \]

\[ \psi_j^{nk} = (-\frac{1}{\bar{p}_j})^n (-\frac{1}{\bar{p}_j + a})^k e^{\bar{\xi}_j} \]

with

\[ \xi_i = \frac{1}{p_i} x_{-1} + p_i x_1 + \frac{1}{p_i - a} t_a + \xi_{i0} \]

\[ \bar{\xi}_j = \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \frac{1}{\bar{p}_j + a} t_a + \bar{\xi}_{j0} \]
Then the following bilinear equations hold

\[
\left( \frac{1}{2} D_{x_1} D_{x_{-1}} - 1 \right) \tau_{n,k} \cdot \tau_{n,k} = -\tau_{n+1,k} \tau_{n-1,k}
\]

\[
(aD_{t_\alpha} - 1) \tau_{n+1,k} \cdot \tau_{n,k} = -\tau_{n+1,k-1} \tau_{n,k+1}
\]

\[
(D_{x_1} (aD_{t_\alpha} - 1) - 2a) \tau_{n+1,k} \cdot \tau_{n,k} = (D_{x_1} - 2a) \tau_{n+1,k-1} \cdot \tau_{n,k+1}
\]

Objective bilinear equations:

\[
(D_s D_y - i\omega D_y + i\kappa D_s)g \cdot f = 0, \quad D_s^2 f \cdot f = \frac{1}{2}\omega^2 (f^2 - |g|^2)
\]
By taking
\[ \bar{p}_j = \frac{1}{p_j}, \quad a = i \]
we have
\[ p_i + \bar{p}_i = \frac{1}{p_i} + \frac{1}{\bar{p}_i} \]
\[ -\frac{\bar{p}_i}{p_i} \left( -\frac{p_i - a}{\bar{p}_i + a} \right)^2 = 1 \]
thus \( \tau_{nk} \) satisfies the reduction conditions
\[ \partial_{x_1} \tau_{nk} = \partial_{x_{-1}} \tau_{nk} \]
\[ \tau_{n-1,k+2} = \tau_{nk}. \]
Then the first bilinear equation becomes
\[ \left( \frac{1}{2} D_{x_1}^2 - 1 \right) \tau_{nk} \cdot \tau_{nk} = -\tau_{n+1,k} \tau_{n-1,k} \]
Moeover, from the other bilinear equations and the above reductions, we have

\[
(D_{x_1}(aD_{t_a} - 1) - 2a) \tau_{n+1,k} \cdot \tau_{nk}
\]
\[
= (D_{x_1} - 2a) \tau_{n+1,k-1} \cdot \tau_{n,k+1} (= \tau_{n+1,k-1})
\]
\[
= -2a \tau_{n+1,k-1} \cdot \tau_{n+1,k-1} (= \tau_{n,k+1})
\]
\[
= 2a(aD_{t_a} - 1) \tau_{n+1,k} \cdot \tau_{nk}
\]

i.e.,

\[
(D_{x_1}(D_{t_a} + i) - 2iD_{t_a}) \tau_{n+1,k} \cdot \tau_{nk} = 0
\]
Reductions to the dCSP equation

By taking $|p_i| = 1$, $\bar{\xi}_{j0} = \xi_{j0}^*$, where * means complex conjugate, we have

$$\tau_{n0}^* = \tau_{-n,0}$$

$$\tau_{n0} = \left| \delta_{ij} + \frac{1}{p_i + p_j^*} \left( - \frac{p_i}{p_j^*} \right)^n e^{\xi_i + \xi_j^*} \right|_{1 \leq i, j \leq N}$$

By defining

$$f = \tau_{00}, g = \tau_{10}$$

we get

$$\left( \frac{1}{2} D_{x_1}^2 - 1 \right) f \cdot f = -gg^*$$

$$(D_{x_1} D_t + iD_{x_1} - 2iD_{t_\alpha}) g \cdot f = 0.$$  

Finally, by setting $t_\alpha = \kappa y$, $2x_1 = \omega s$, the above bilinear equations are converted into

$$\left( D_s^2 - \frac{\omega^2}{2} \right) f \cdot f = -\frac{\omega^2}{2} gg^*$$

$$(D_{y} D_{t_\alpha} + \omega D_{x_1} + 2iD_{y} D_{t_\alpha}) g \cdot f = 0.$$
Summary for the focusing and defocusing CSP equation

- The bright soliton solution to the focusing CSP equation can be obtained from the reduction of the two-component KP hierarchy or from the Darboux transformation.

- The rogue wave solution to the focusing CSP equation can be obtained from the Darboux transformation, we are working on the higher order rogue wave solutions by Hirota’s bilinear method.

- The dark soliton solution to the defocusing CSP equation can be obtained from the reduction of the one-component KP hierarchy or from the Darboux transformation.
Theorem

Bilinear equations

\[
\begin{align*}
\frac{1}{a} D_s (g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) &= g_{k+1} f_k + g_k f_{k+1}, \\
D_s^2 f_k \cdot f_k &= \frac{1}{2} g_k g_k^*.
\end{align*}
\]

give semi-discrete complex SP equation

\[
\begin{align*}
\frac{d}{dt}(q_{k+1} - q_k) &= \frac{1}{2} (x_{k+1} - x_k)(q_{k+1} + q_k), \\
\frac{d}{dt}(x_{k+1} - x_k) &= -\frac{1}{2} (|q_{k+1}|^2 - |q_k|^2).
\end{align*}
\]

through transformations

\[
q_k = \frac{g_k}{f_k}, \quad x_k = 2ka - 2(\ln f_k)_s.
\]
Multi-soliton solution:

\[ f_k = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_k = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_1 & 0 \end{vmatrix}, \]

where the elements defined respectively by

\[ a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{-1})} e^{\xi_i + \bar{\xi}_j}, \quad b_{ij} = \frac{\alpha_i^* \alpha_j}{2(p_j^{-1} + p_i^{-1})} \]

\[ e^{\xi_i} = \left( \frac{1 + \alpha p_i}{1 - \alpha p_i} \right)^k \exp\left( \frac{1}{p_i} s + \xi_{i0} \right), \quad e^{\xi_j^*} = \left( \frac{1 + \alpha p_j^*}{1 - \alpha p_j^*} \right)^k \exp\left( \frac{1}{p_i^*} s + \bar{\xi}_{j0} \right). \]
Lax pair to the semi-discrete CSP equation

$$\Psi_{k+1} = U_k \Psi_k, \quad \Psi_{k,t} = V_k \Psi_k,$$

with

$$U_k = \begin{pmatrix} 1 - i\lambda \delta_k & -i\lambda(q_{k+1} - q_k) \\ -i\lambda(q^*_k - q^*_k) & 1 + i\lambda \delta_k \end{pmatrix}$$

$$V_k = \begin{pmatrix} \frac{i}{4\lambda} & -\frac{1}{2} q_k \\ \frac{1}{2} q^*_k & -\frac{i}{4\lambda} \end{pmatrix}$$

The compatibility condition $$d U_k / dt + U_k V_k - V_{k+1} U_k = 0$$ gives the semi-discrete CSP equation.
Bilinear equations

\[
\begin{align*}
&g_{k+1}^{l+1}f_k^l - g_k^{l+1}f_{k+1}^l - g_{k+1}^lf_{k+1}^l + g_k^lf_{k+1}^l \\
&= ab(g_{k+1}^{l+1}f_k^l + g_k^{l+1}f_{k+1}^l + g_{k+1}^lf_{k+1}^l + g_k^lf_{k+1}^l)
\end{align*}
\]

\[f_{k+1}^{l+1}f_{k+1}^{l-1} - f_k^{l+1}f_k^l = b^2g_k^l\bar{g}_k^l\]

give the fully discrete complex SP equation

\[
\begin{align*}
&(1 - ab)(q_k^l + q_{k+1}^{l+1}) = (1 + ab)(q_{k+1}^{l+1} + q_k^l) (1 + (\delta_k^l - 2a)b) \\
&\frac{1 + (\delta_k^l - 2a)b}{1 + (\delta_k^{l-1} - 2a)b} = \frac{1 + b^2q_k^l\bar{q}_k^l}{1 + b^2q_{k+1}^l\bar{q}_{k+1}^l}
\end{align*}
\]

through transformations

\[
q_k^l = \frac{g_k^l}{f_k^l}, \quad \delta_k^l = 2a + \frac{1}{b} \left( \frac{f_{k+1}^{l+1}f_k^l}{f_{k+1}^l f_k^{l+1}} - 1 \right).
\]
Conclusion and further topics

- We have proposed a focusing and defocusing complex short pulse equation to describe the propagation of ultra-short pulse in optical fibers.
- The multi-bright and multi-dark soliton solutions are obtained from the reductions of the KP hierarchies.
- The soliton, breather and rogue wave solutions are constructed via the Darboux transformation.

Further topics:

1. Physical applications
2. Self-adaptive moving method based on integrable discretizations
3. Studies for the coupled CSP equation
We have proposed a focusing and defocusing complex short pulse equation to describe the propagation of ultra-short pulse in optical fibers.

The multi-bright and multi-dark soliton solutions are obtained from the reductions of the KP hierarchies.

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- **Further topic 1**: Physical applications
- **Further topic 2**: Self-adaptive moving method based on integrable discretizations
- **Further topic 3**: Studies for the coupled CSP equation