Numerical methods for some integrable PDEs

Part 1:

Soliton interaction of Davey-Stewartson II system

(Part 2: Self-adaptive moving mesh schemes)

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Part 1: Soliton interaction of Davey-Stewartson II system

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Motivation & Background

- Detailed studies of line soliton interactions of the KP II equation which describes 2-dimensional weak nonlinear shallow water waves (Kodama and his collaborators, 2003〜)
- Line soliton interactions of the Davey-Stewartson (DS) system which describes 2-dimensional weak nonlinear deep water waves haven't been studied much.
- Develop a numerical method to study DS line soliton interactions.
- Study dark line soliton interactions of the defocusing DS II system(hyperbolic-elliptic) by using the theory based on tau-functions and numerics.

Davey-Stewartson(DS) System

<DS system> (Davey & Stewartson 1974)

$$iu_t - \sigma_1 u_{xx} + u_{yy} - \sigma_2 |u|^2 u - 4\sigma_1 u \phi_x = 0$$
$$\sigma_1 \phi_{xx} + \phi_{yy} + \frac{1}{2}\sigma_2 (|u|^2)_x = 0$$

$\sigma_1 =$
- +1 —— DS II (hyperbolic-elliptic type)
- -1 —— DS I (elliptic-hyperbolic)

σ_2 determines focusing and defocusing.

In this talk, we focus on the defocusing DS II system.

$\sigma_1 = 1, \sigma_2 = 1$

Line soliton solutions of the DS system

Find line soliton solutions for DS system by using Hirota bilinear method.

$$iu_t - \sigma_1 u_{xx} + u_{yy} - \sigma_2 |u|^2 u - 4\sigma_1 u\phi_x = 0$$

$$\sigma_1 \phi_{xx} + \phi_{yy} + \frac{1}{2}\sigma_2 (|u|^2)_x = 0$$

$$u = \rho_0 \frac{g}{f} e^{i(kx+ly-\omega t+\xi^0)}$$

$$\phi = (\log f)_x$$

Bilinear equations

$$(iD_t - 2ik\sigma_1 D_x + 2ilD_y - \sigma_1 D_x^2 + D_y^2) g \cdot f = 0$$

$$(\sigma_1 D_x^2 + D_y^2 - \sigma_2 \rho_0^2) f \cdot f + \sigma_2 \rho_0^2 g g^* = 0$$

$$D_x^m D_t^n f \cdot g = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n f(x,t) g(x',t')|_{x'=x,t'=t}$$

1-soliton solution of the DSII system

$$f = 1 + e^{px+qy-\Omega t+\theta^0}, \quad g = 1 + \alpha e^{px+qy-\Omega t+\theta^0}, \quad g^* = 1 + \alpha^* e^{px+qy-\Omega t+\theta^0}$$

$$\alpha = \frac{2\sigma_1 kp - 2lq + \imath(\sigma_1 p^2 - q^2) + \Omega}{2\sigma_1 kp - 2lq - \imath(\sigma_1 p^2 - q^2) + \Omega}$$

$$\Omega = -2\sigma_1 kp + 2lq + \frac{\sqrt{(\sigma_1 p^2 - q^2)(p^2 - q^2)(p^2 + q^2)(2\sigma_2\rho_0^2 - \sigma_1 p^2 - q^2)}}{\sigma_1 p^2 + q^2}$$

$$\omega = -\sigma_1 k^2 + l^2 + \sigma_2 \rho_0^2$$

$$u = \rho_0 e^{i(kx+ly-\omega t+\xi^{(0)})} \frac{1 + \alpha e^{px+qy-\Omega t+\theta^0}}{1 + e^{px+qy-\Omega t+\theta^0}}$$

$$|u|^2 = \rho_0^2 - \rho_0^2 \frac{p^2+q^2}{2} \mathrm{sech}^2 \frac{px+qy-\Omega t+\theta^0}{2}$$

Depth $\rho_0^2 - \dfrac{p^2+q^2}{2}$

Dark soliton

$$\phi_x = \frac{p^2}{4} \mathrm{sech}^2 \frac{px+qy-\Omega t+\theta^0}{2}$$

Amplitude $\dfrac{p^2}{4}$

2-soliton solution of the DSII system

$$u = \rho_0 \frac{g}{f} e^{i(kx+ly-\omega t+\xi^0)} \qquad \phi = (\log f)_x$$

$$f = 1 + e^{\theta_1} + e^{\theta_2} + A_{12} e^{\theta_1+\theta_2}$$

$$g = 1 + \alpha_1 e^{\theta_1} + \alpha_2 e^{\theta_2} + \alpha_1 \alpha_2 A_{12} e^{\theta_1+\theta_2}$$

$$g^* = 1 + \alpha_1^* e^{\theta_1} + \alpha_2^* e^{\theta_2} + \alpha_1^* \alpha_2^* A_{12} e^{\theta_1+\theta_2}$$

$$\alpha_i = \frac{2\sigma_1 k p_i - 2l q_i + \imath(\sigma_1 p_i^2 - q_i^2) + \Omega_i}{2\sigma_1 k p_i - 2l q_i - \imath(\sigma_1 p_i^2 - q_i^2) + \Omega_i} \;,\; \theta_i = p_i x + q_i y - \Omega_i t + \theta_i^{(0)}$$

$$\omega = -\sigma_1 k^2 + l^2 + \sigma_2 \rho_0^2$$

$$\Omega_1 = -2(\sigma_1 k p_1 - l q_1) \pm \frac{\sqrt{-(q_1^2 - \sigma_1 p_1^2)^2 (\sigma_1 p_1^2 + q_1^2)(q_1^2 - 2\rho_0^2 \sigma_2 + \sigma_1 p_1^2)}}{\sigma_1 p_1^2 + q_1^2}$$

$$\Omega_2 = -2(\sigma_1 k p_2 - l q_2) \pm \frac{\sqrt{-(q_2^2 - \sigma_1 p_2^2)^2 (\sigma_1 p_2^2 + q_2^2)(q_2^2 - 2\rho_0^2 \sigma_2 + \sigma_1 p_2^2)}}{\sigma_1 p_2^2 + q_2^2}$$

$\sigma_1 = 1, \sigma_2 = 1$ (DSII defocusing)

$$A_{12} = \frac{\sqrt{(p_1^2-q_1^2)^2(p_1^2+q_1^2)(2\rho_0^2-p_1^2-q_1^2)(p_2^2-q_2^2)^2(p_2^2+q_2^2)(2\rho_0^2-p_2^2-q_2^2)}+(p_1^2-q_1^2)(p_2^2-q_2^2)((p_1^2+q_1^2)(p_2^2+q_2^2)-2\rho_0^2(p_1p_2+q_1q_2))}{\sqrt{(p_1^2-q_1^2)^2(p_1^2+q_1^2)(2\rho_0^2-p_1^2-q_1^2)(p_2^2-q_2^2)^2(p_2^2+q_2^2)(2\rho_0^2-p_2^2-q_2^2)}-(p_1^2-q_1^2)(p_2^2-q_2^2)((p_1^2+q_1^2)(p_2^2+q_2^2)+2\rho_0^2(p_1p_2+q_1q_2))}$$

(when $\pm$ in Ω_1, Ω_2 are same signs)

or

$$A_{12} = \frac{\sqrt{(p_1^2-q_1^2)^2(p_1^2+q_1^2)(2\rho_0^2-p_1^2-q_1^2)(p_2^2-q_2^2)^2(p_2^2+q_2^2)(2\rho_0^2-p_2^2-q_2^2)}-(p_1^2-q_1^2)(p_2^2-q_2^2)((p_1^2+q_1^2)(p_2^2+q_2^2)-2\rho_0^2(p_1p_2+q_1q_2))}{\sqrt{(p_1^2-q_1^2)^2(p_1^2+q_1^2)(2\rho_0^2-p_1^2-q_1^2)(p_2^2-q_2^2)^2(p_2^2+q_2^2)(2\rho_0^2-p_2^2-q_2^2)}+(p_1^2-q_1^2)(p_2^2-q_2^2)((p_1^2+q_1^2)(p_2^2+q_2^2)+2\rho_0^2(p_1p_2+q_1q_2))}$$

(when $\pm$ in Ω_1, Ω_2 are different signs)

Remark:

When a soliton inclined 45 degree from y-axis appears

(i.e., $p_1 = \pm q_1, p_2 = \pm q_2$),

A_{12} changes its property.

KP equation

$$(4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

$$u(x,y,t) = 2(\log \tau(x,y,t))_{xx}$$

$$(4D_xD_t + D_x^4 + 3D_y^2)\tau \cdot \tau = 0$$

2-solution

$$\tau = 1 + e^{\theta_1} + e^{\theta_2} + B_{12}e^{\theta_1+\theta_2}$$

$$B_{12} = \frac{(K_1^x - K_2^x)^2 - (\tan\psi_1 - \tan\psi_2)}{(K_1^x + K_2^x)^2 - (\tan\psi_1 - \tan\psi_2)}$$

$$\theta_j = K_j^x + K_j^y - \Omega_j t + \theta_j^0$$

$$= K_j^x(x + y\tan\psi_j - \frac{1}{4}((K_j^x)^2 + 3\tan\psi_j)t + \frac{\theta_j^0}{K_j^x}) \quad (j = 1,2)$$

$$\tan\psi_j = \frac{K_j^y}{K_j^x}, \quad K_j^x = \sqrt{2A_j}$$

dispersion relation

$$D_{KP}(\Omega_j, \mathbf{K}_j) := -4\Omega_j K_j^x + (K_j^x)^4 + 3(K_j^y)^2 = 0$$

2-soliton solution of the KP equation

$$\tau = 1 + e^{\theta_1} + e^{\theta_2} + B_{12} e^{\theta_1 + \theta_2} \qquad \theta_i = K_i^x x + K_i^y y - \Omega_i t$$

$$B_{12} = -\frac{(K_1^x - K_2^x)^4 + 3(K_1^y - K_2^y)^2 - 4(K_1^x - K_2^x)(\Omega_1 - \Omega_2)}{(K_1^x + K_2^x)^4 + 3(K_1^y + K_2^y)^2 - 4(K_1^x + K_2^x)(\Omega_1 + \Omega_2)}$$
$$= -\frac{D_{KP}(\Omega_1 - \Omega_2, K_1^x - K_2^x, K_1^y - K_2^y)}{D_{KP}(\Omega_1 + \Omega_2, K_1^x + K_2^x, K_1^y + K_2^y)}$$

$$D_{KP}(\Omega_j, \mathbf{K}_j) := -4\Omega_j K_j^x + (K_j^x)^4 + 3(K_j^y)^2 = 0$$

B_{12} must be non-negative for the existence of a non-singular solution.

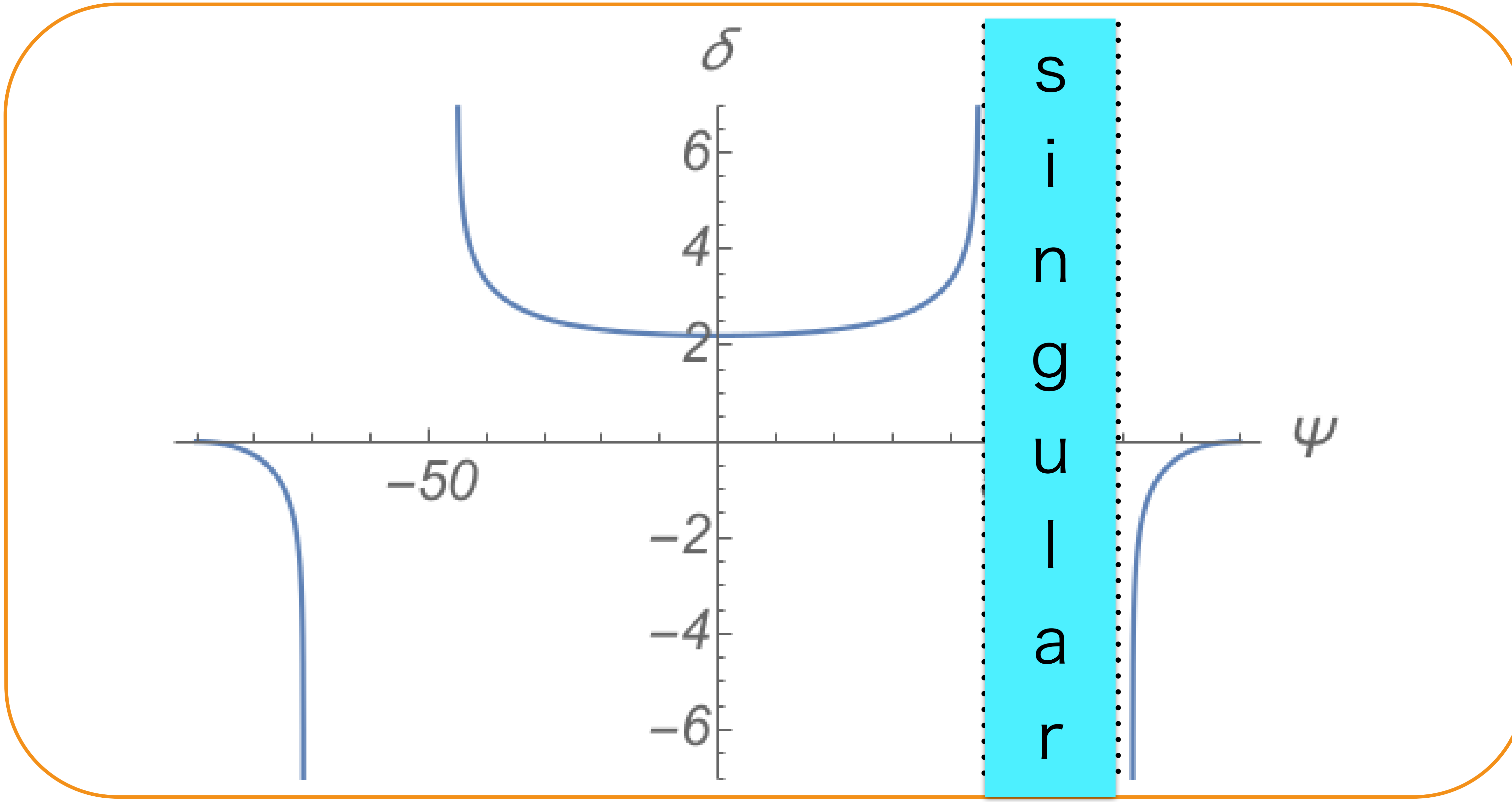
1. $(\tan\psi_1 - \tan\psi_2)^2 < (K_1^x - K_2^x)^2 < (K_1^x + K_2^x)^2$
   $\Rightarrow \; 0 < B_{12} < 1$

2. $(K_1^x - K_2^x)^2 < (\tan\psi_1 - \tan\psi_2)^2 < (K_1^x + K_2^x)^2 \Rightarrow$ $\boxed{B_{12} < 0}$

3. $(K_1^x - K_2^x)^2 < (K_1^x + K_2^x)^2 < (\tan\psi_1 - \tan\psi_2)^2 \Rightarrow \; 0 < 1 < B_{12}$

In 2, the above 2-soliton solution becomes singular.

Angle dependency of KP 2-soliton solution



$$\delta = -\log B_{12}$$

Horizontal axis is the angle of 2 solitons.

(Amplitudes of solitons are constant.)

Wronskian form of KP line solitons

$$\tau = \begin{vmatrix} f_1^0 & \cdots & f_N^0 \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix}, \qquad f_i^{(n)} := \frac{\partial^n}{\partial x^n} f_i$$

$$\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2}, \qquad \frac{\partial f_i}{\partial t} = -\frac{\partial^3 f_i}{\partial x^3}$$

$$f_i(x,y,t) = \sum_{j=1}^{M} a_{ij} E_j(x,y,t), \qquad E_j(x,y,t) := e^{k_j x + k_j^2 y - k_j^3 t + \theta_j^0}$$

$$i = 1, \cdots, N < M$$

$$\Longrightarrow \quad \tau(x,y,t) = \sum_{1 \le m_1 \le \cdots \le m_N \le M} A(m_1, \cdots, m_N) W_r(E_{m_1}, \cdots, E_{m_N})$$

Binet-Cauchy formula

$$W_r(E_{m_1}, \cdots, E_{m_N}) = \prod_{1 \le s \le r \le N} (k_{m_r} - k_{m_s}) \prod_{j=1}^{N} E_{m_j}$$

$$E_j = e^{(k_j x + k_j^2 y - k_j^3 t)}, \quad k_1 < k_2 < \cdots < k_M$$

$A(m_1, \cdots, m_N)$: a determinant of selecting columns $m_1, \cdots, m_N$ of $A = (a_{i,j})_{1 \le i \le N, 1 \le j \le M}$

A-matrix determines non-negativity of tau-function and types of soliton interactions

KP equation
P-type 2-soliton solution

2×2 matrix (a, b are positive)

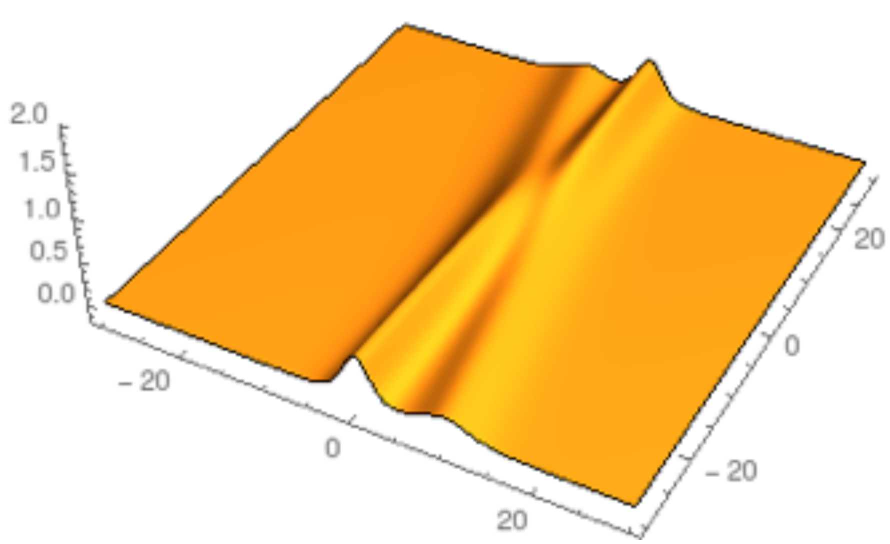
$$A = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix}$$

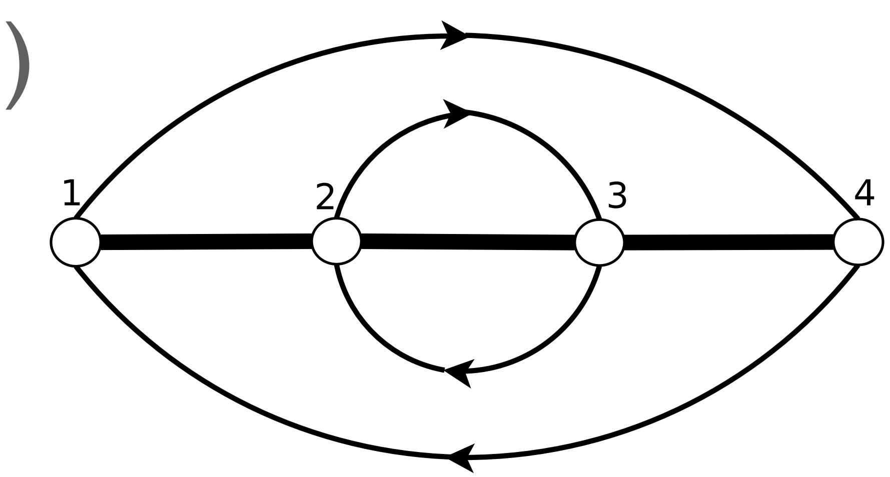
$$\tau = (k_2 - k_1)e^{\theta(1,2)} + (k_3 - k_1)ae^{\theta(1,3)} + (k_4 - k_2)be^{\theta(2,4)} + (k_4 - k_3)abe^{\theta(3,4)}$$

$$(\theta(i,j) = \theta_i + \theta_j)$$

Asypmtotic solitons: [1,4]-soliton and [2,3]-soliton.

(4 3 2 1)





chord diagram

KP equation
T-type 2-soliton solution

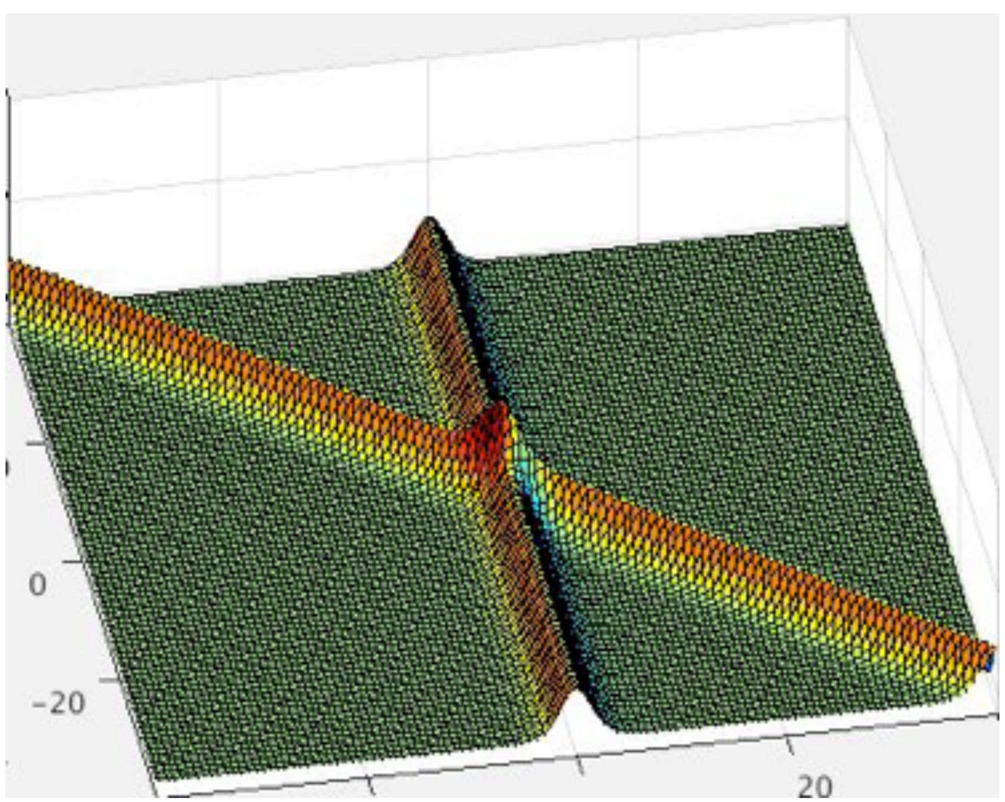
2×2 matirx (a,b,c,d are positive, $ad - bc > 0$)

$$A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix}$$

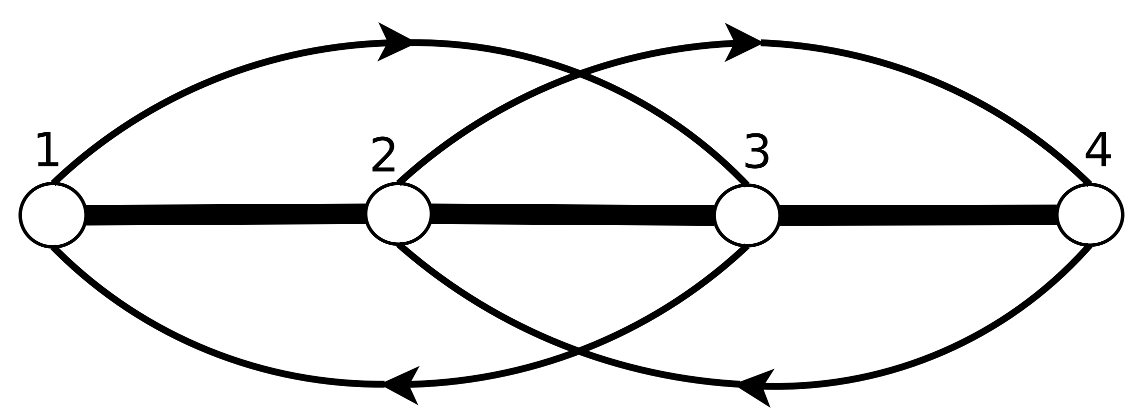
6 terms

$$\tau = (k_2 - k_1)e^{\theta(1,2)} + (k_3 - k_1)ae^{\theta(1,3)} + (k_4 - k_1)be^{\theta(1,4)} + (k_3 - k_2)ce^{\theta(2,3)} + (k_4 - k_2)de^{\theta(2,4)} + (k_4 - k_3)(ad - bc)e^{\theta(3,4)}$$

Asymptotic solitons: [1,3]-soliton and [2,4]-soliton.



(3 4 1 2)



This appears when 2-soliton solution by Hirota method becomes singular.

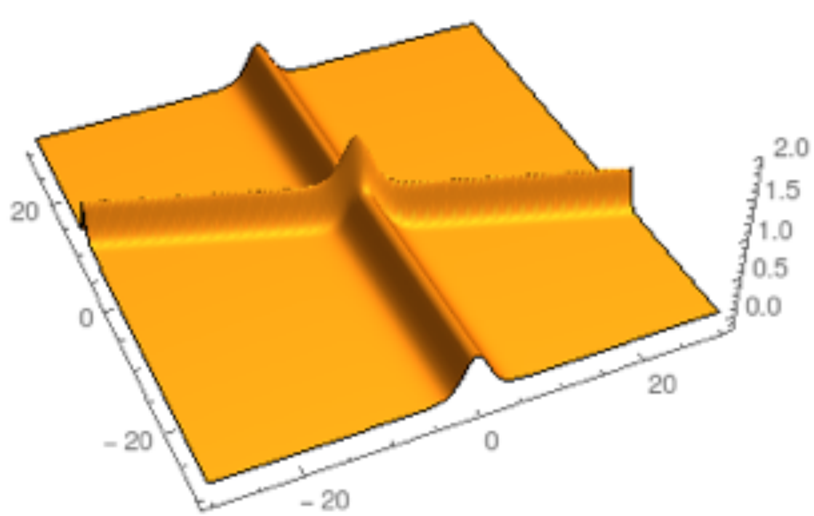
KP equation
O-type 2-soliton

2×2 matrix (a, b are positive)

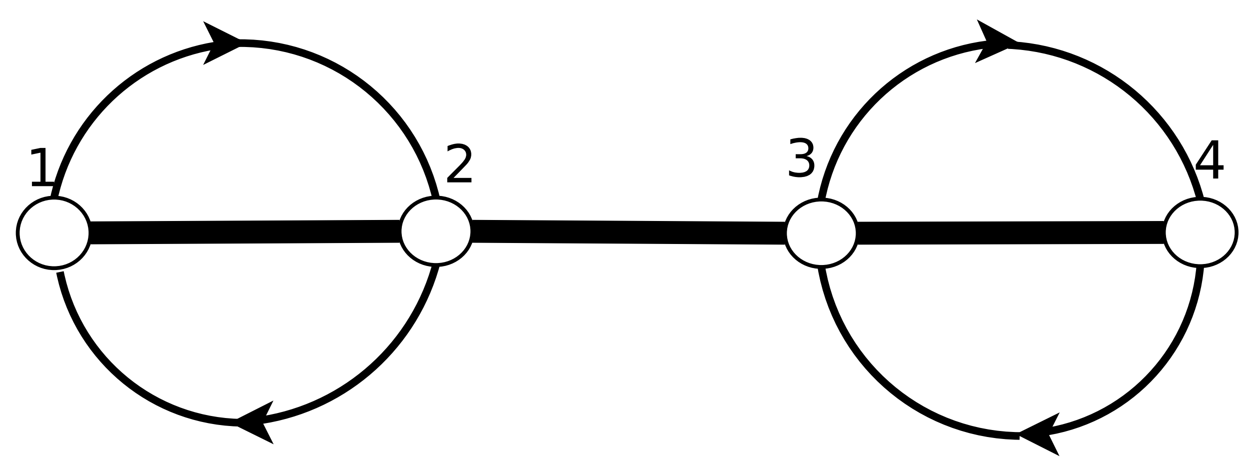
$$A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}$$

$$\tau = (k_3 - k_1)e^{\theta(1,3)} + (k_4 - k_1)be^{\theta(1,4)} + (k_3 - k_2)ae^{\theta(2,3)} + (k_4 - k_2)abe^{\theta(2,4)}$$

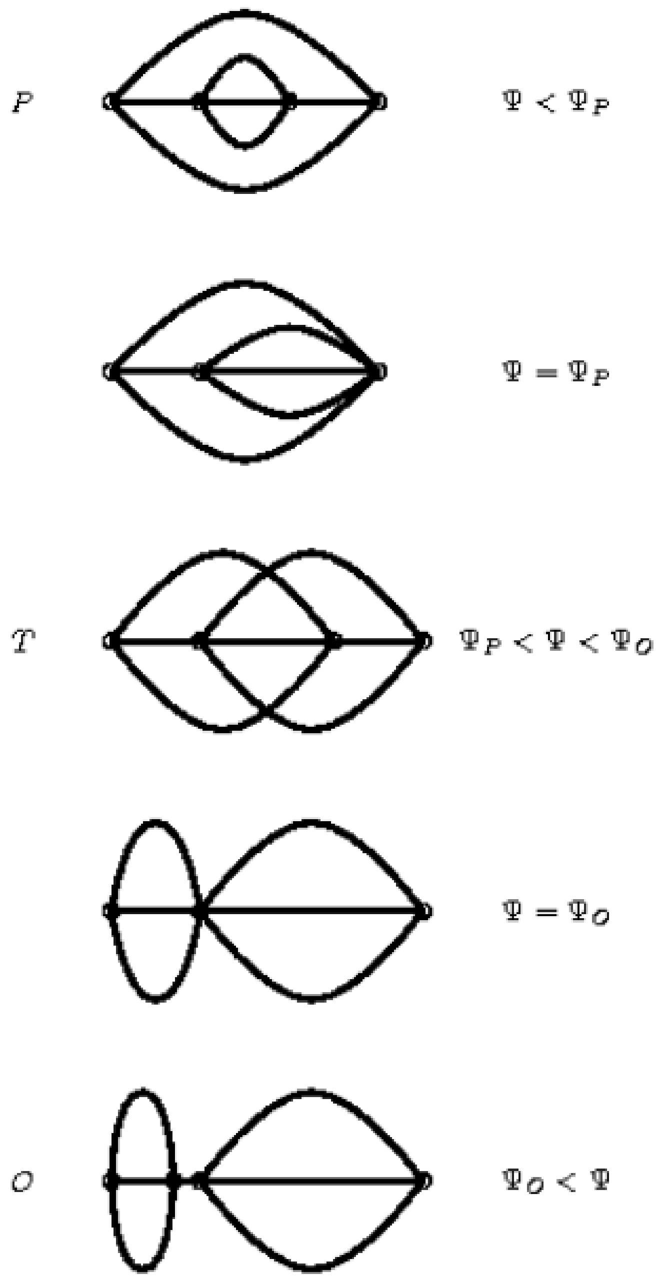
Asymptotic solitons: [1,3]-soliton and [2,4]-soliton

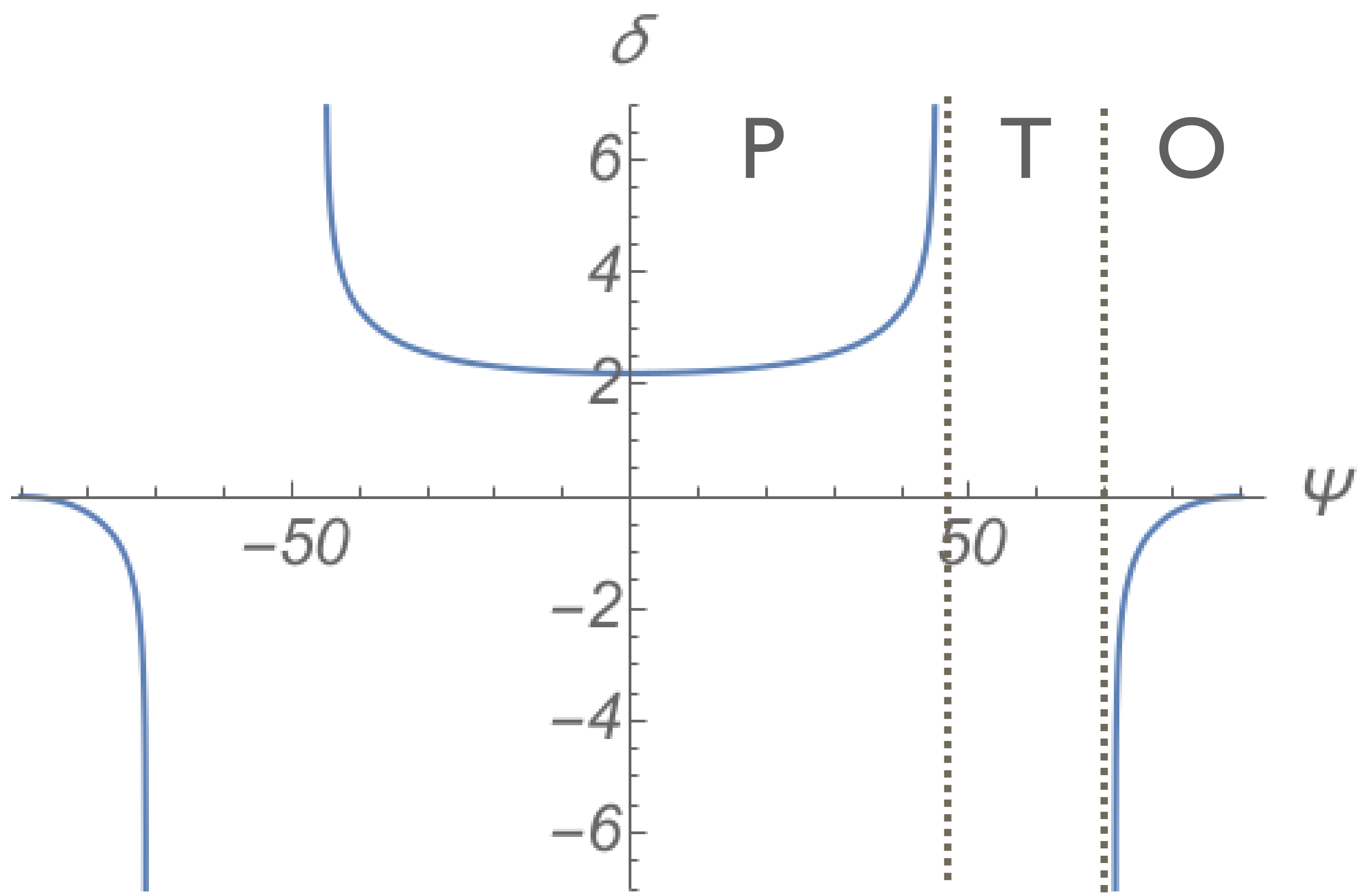


(2 1 4 3)

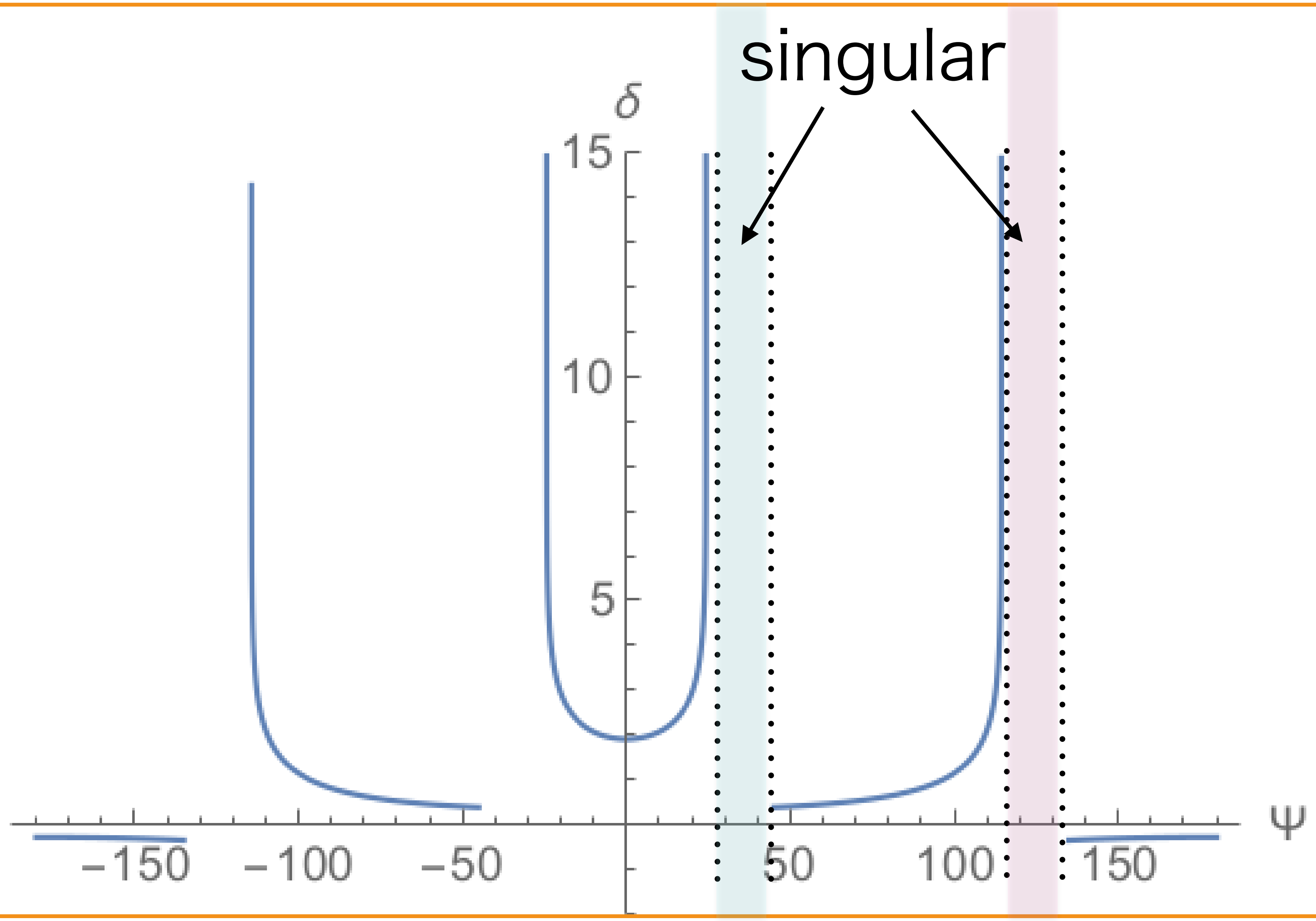


KP SOLITON INTERACTION





Angle dependency of DS II 2-soliton solution



$\delta = -\log A_{12}$

$p_1 = 2, q_1 = 0, p_2 = \cos\Psi, q_2 = \sin\Psi$

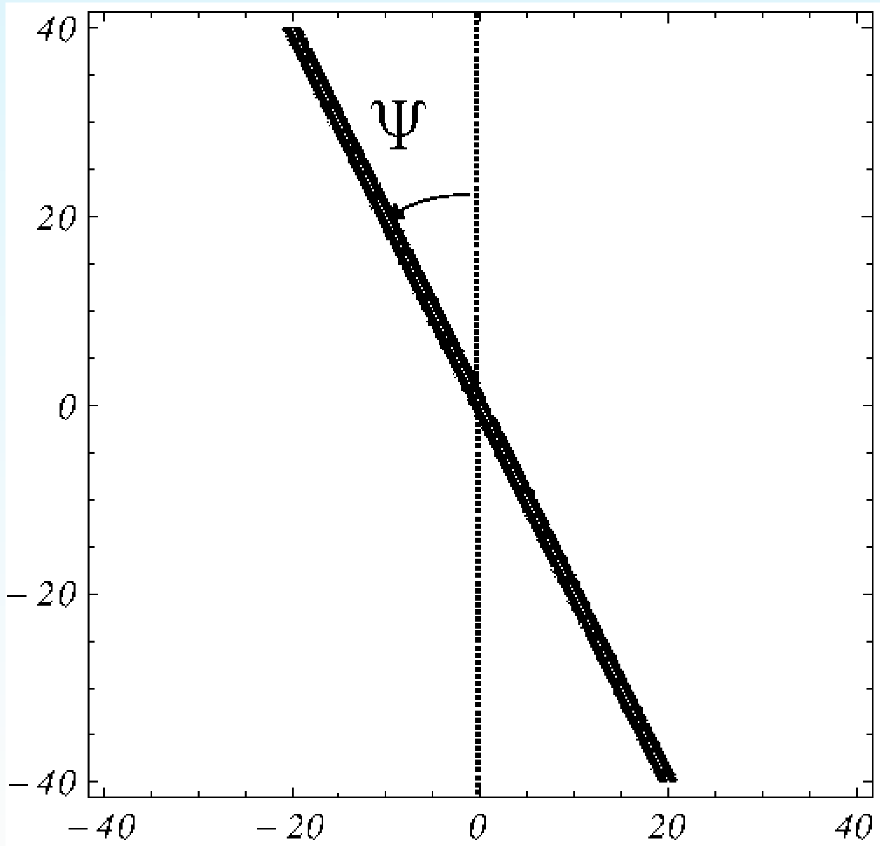
Horizontal axis is the angle between 2 solitons.

(Amplitudes of solitons are constant.)

DSII Soliton Interaction

★ There exists two regions in which DS II 2-soliton solution becomes singular. In these regions, what kind of soliton interactions appears?

★ Investigate 2-soliton interactions by numerics when the angle between 2 solitons is increased.



A numerical method for DSII system ：Split-Step Fourier method

<Linear part>

$$iu_t - \sigma_1 u_{xx} + u_{yy} = 0 \quad \cdots \quad ①$$

<Nonlinear part>

$$\left(\begin{array}{ll} iu_t - \sigma_2 |u|^2 u - 4\sigma_1 u \phi_x = 0 & \cdots \quad ② \\ \sigma_1 \phi_{xx} + \phi_{yy} + \frac{1}{2}\sigma_2 (|u|^2)_x = 0 & \cdots \quad ③ \end{array} \right.$$

White & Weideman 1994 : Numerical methods for DS2 lumps and DS1 dromions

White & Weideman's Algorithm

<Linear part>

$$u_{jk} = \sum_m \sum_n a_{mn} e^{i(\mu_m x_j + \nu_n y_k)}$$

Fourier transform of ① $\Rightarrow$ $i\dfrac{da_{mn}}{dt} - (\mu^2 - \nu^2) a_{mn} = 0$

$$a_{mn}(t + \Delta t) = \exp\left(-i(\mu^2 - \nu^2)\right) a_{mn}(t)$$

<Nonlinear part>

$$|u_{jk}|^2 = \sum_m \sum_n b_{mn} e^{i(\mu_m x_j + \nu_n y_k)} \qquad \phi_{jk} = \sum_m \sum_n c_{mn} e^{i(\mu_m x_j + \nu_n y_k)}$$

Fourier transform of ③ $\Rightarrow$ $(\mu_m^2 + \nu_n^2) c_{mn} - \dfrac{1}{2} i \mu_m b_{mn} = 0$ $\Rightarrow$ $c_{mn} = \dfrac{i\mu_m b_{mn}}{2(\mu_m^2 + \nu_n^2)}$

$$(\phi_{jk})_x = \sum_m \sum_n i\mu_m c_{mn} e^{i(\mu_m x_j + \nu_n y_k)}$$

② $\Rightarrow$ $u(t + \Delta t) = u(t) \exp\left(-i(|u|^2 + 4\phi_x)\right)$

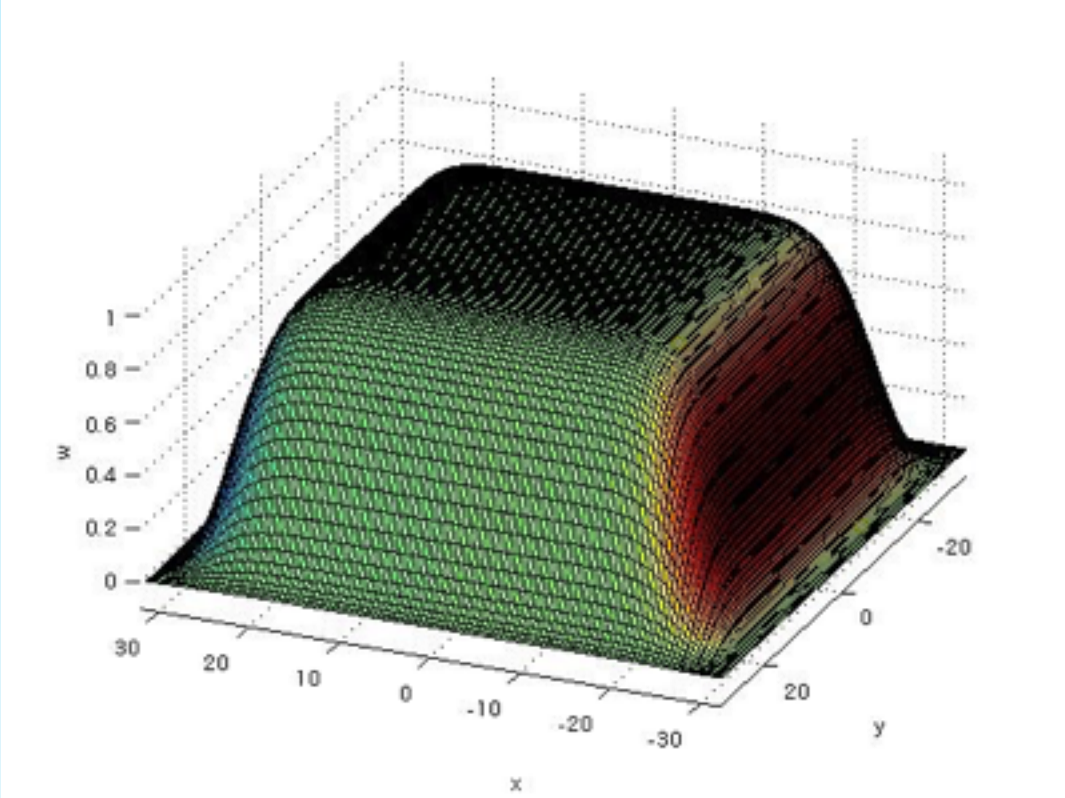
Window method

★ A technique to adjust boundary.

★ Introduce a window function.

★ Example of window functions

$$w(x,y) = 10^{-a^n|2\frac{x-Lx}{-Lx-Lx}-1|^n} * 10^{-a^n|2\frac{y-Ly}{-Ly-Ly}-1|^n}$$



★ Height 1 around center, quickly decreasing at boundary.

★ Dewindowing:
for exact solution v, numerical solution u'

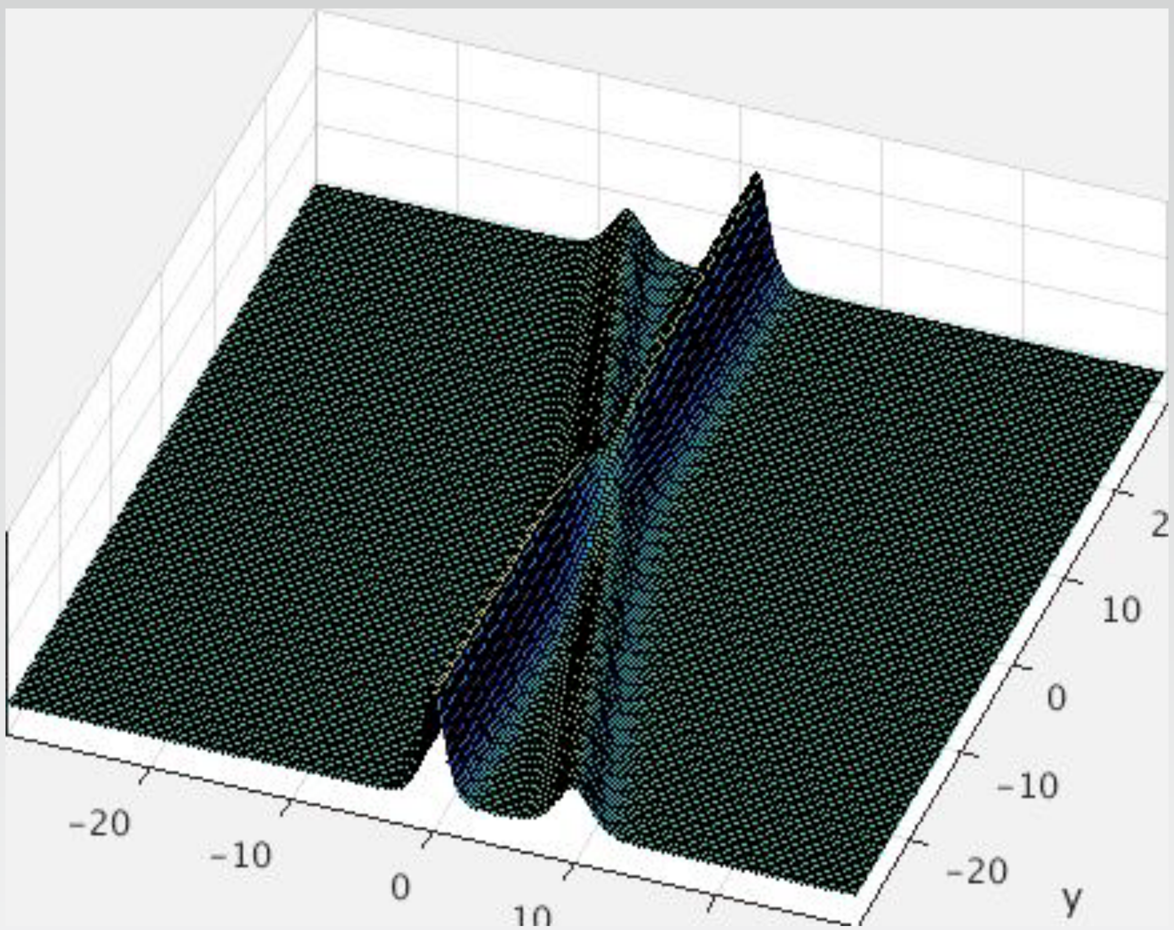
$$u = (1-w)v + wu'$$

★Our numerics use a=1.1, n=30.

2-soliton

Angle:15 degree, $A_{12} = 0.0136$

ϕ_x



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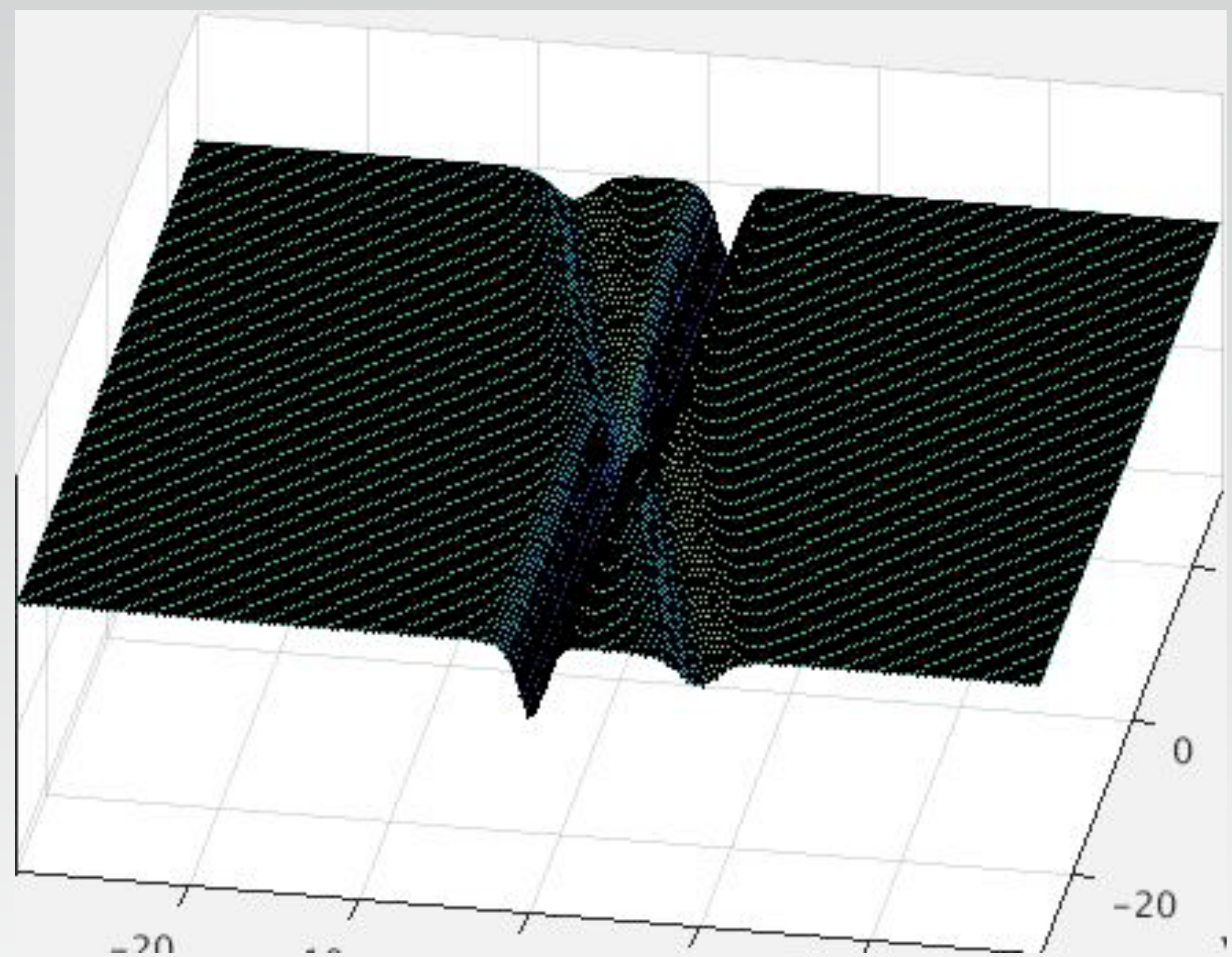
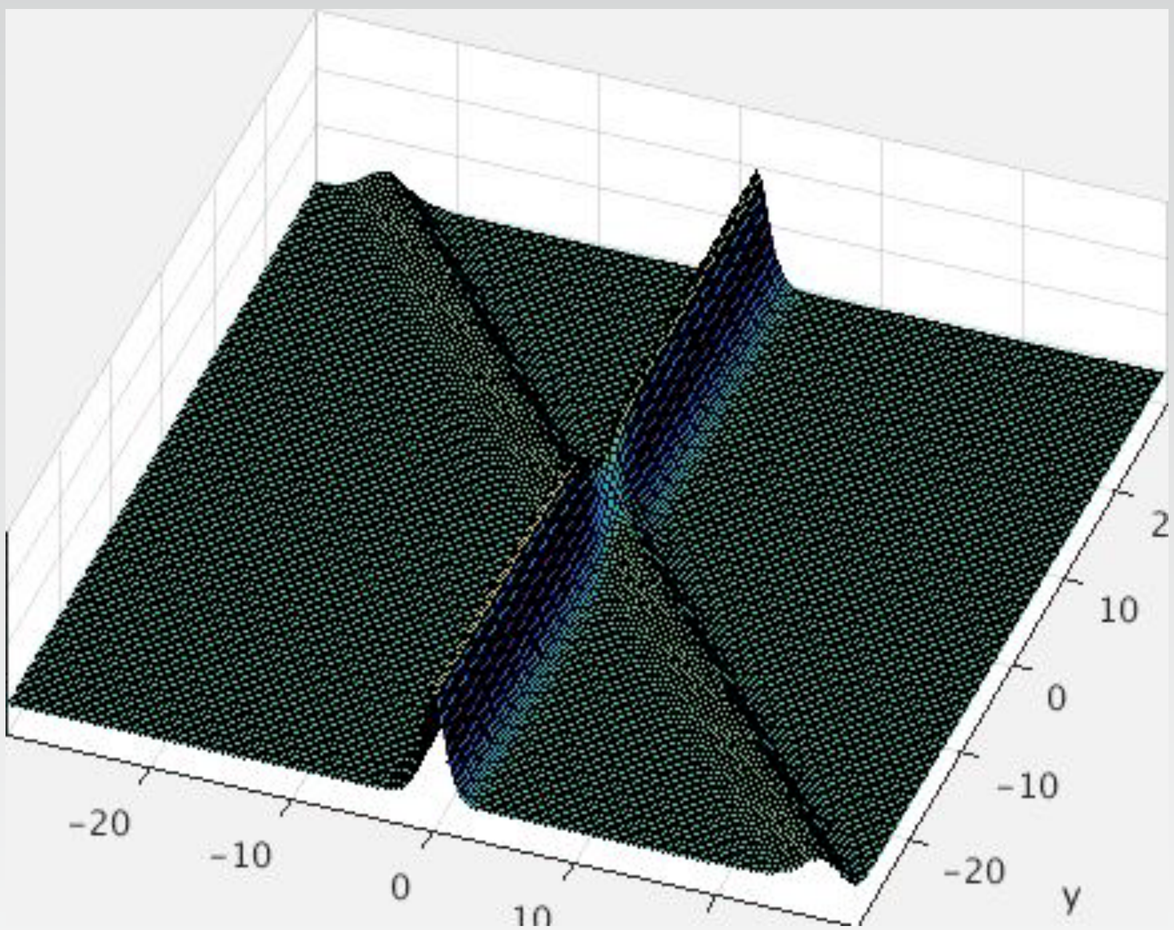


2-soliton

Angle: 40 degree $\quad A_{12} = -0.4037$

ϕ_x



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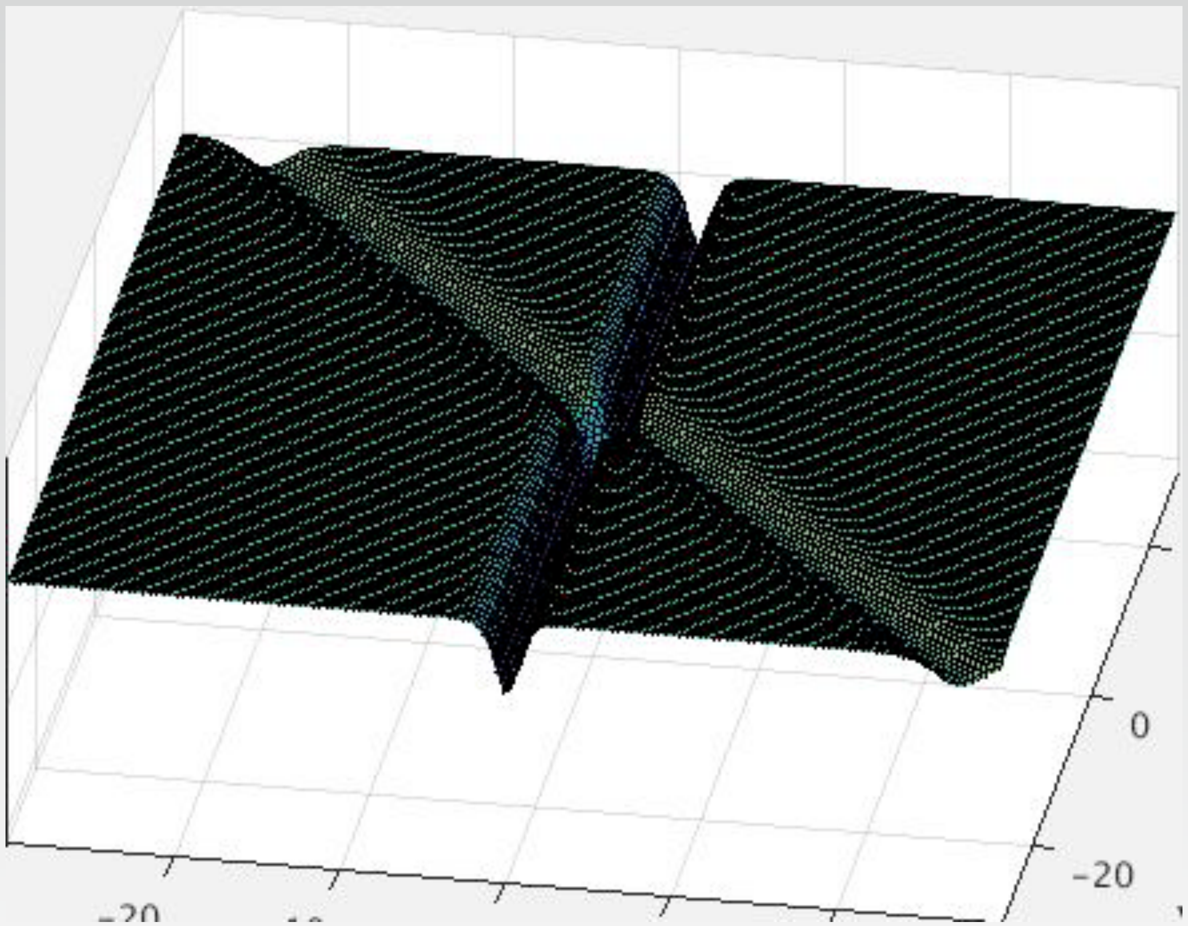
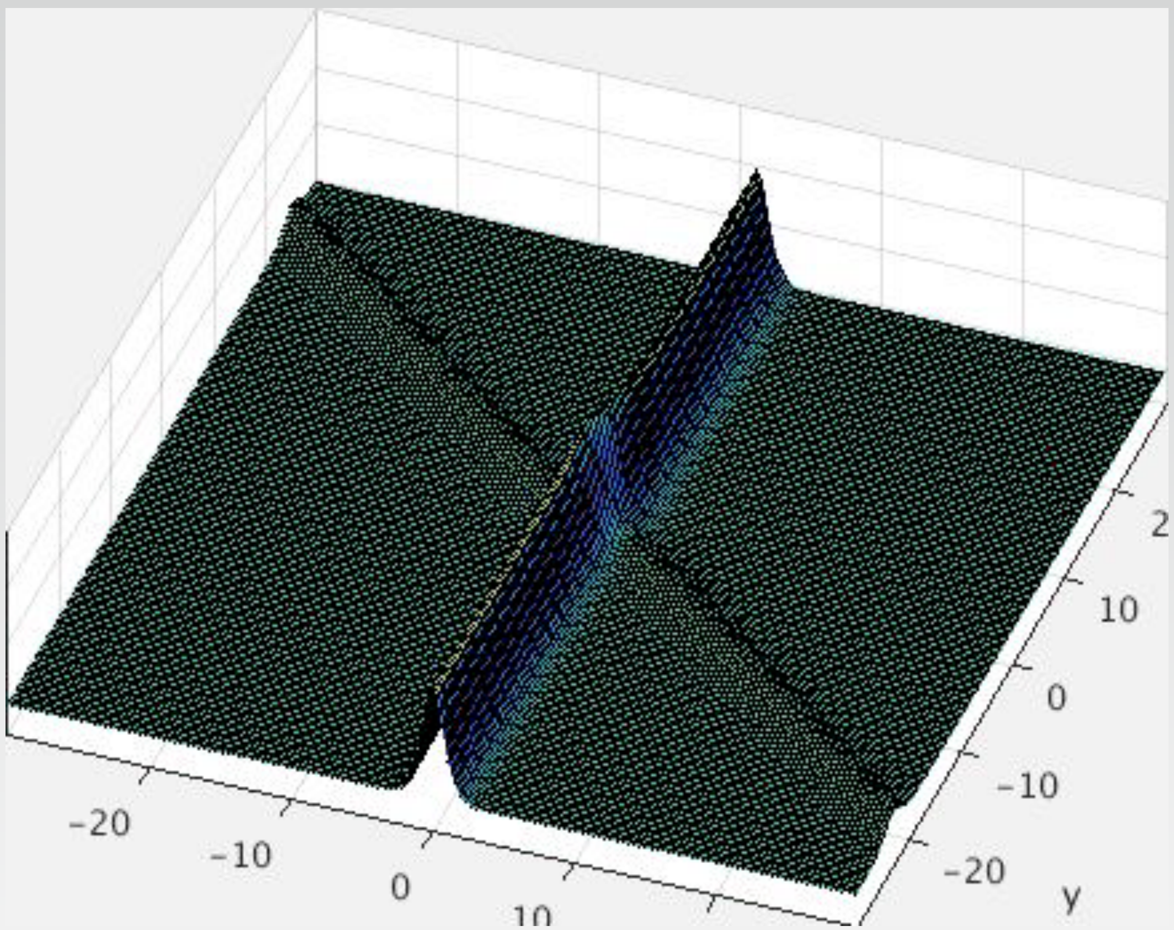


2-soliton

Angle: 50 degree $A_{12} = 0.5927$

ϕ_x



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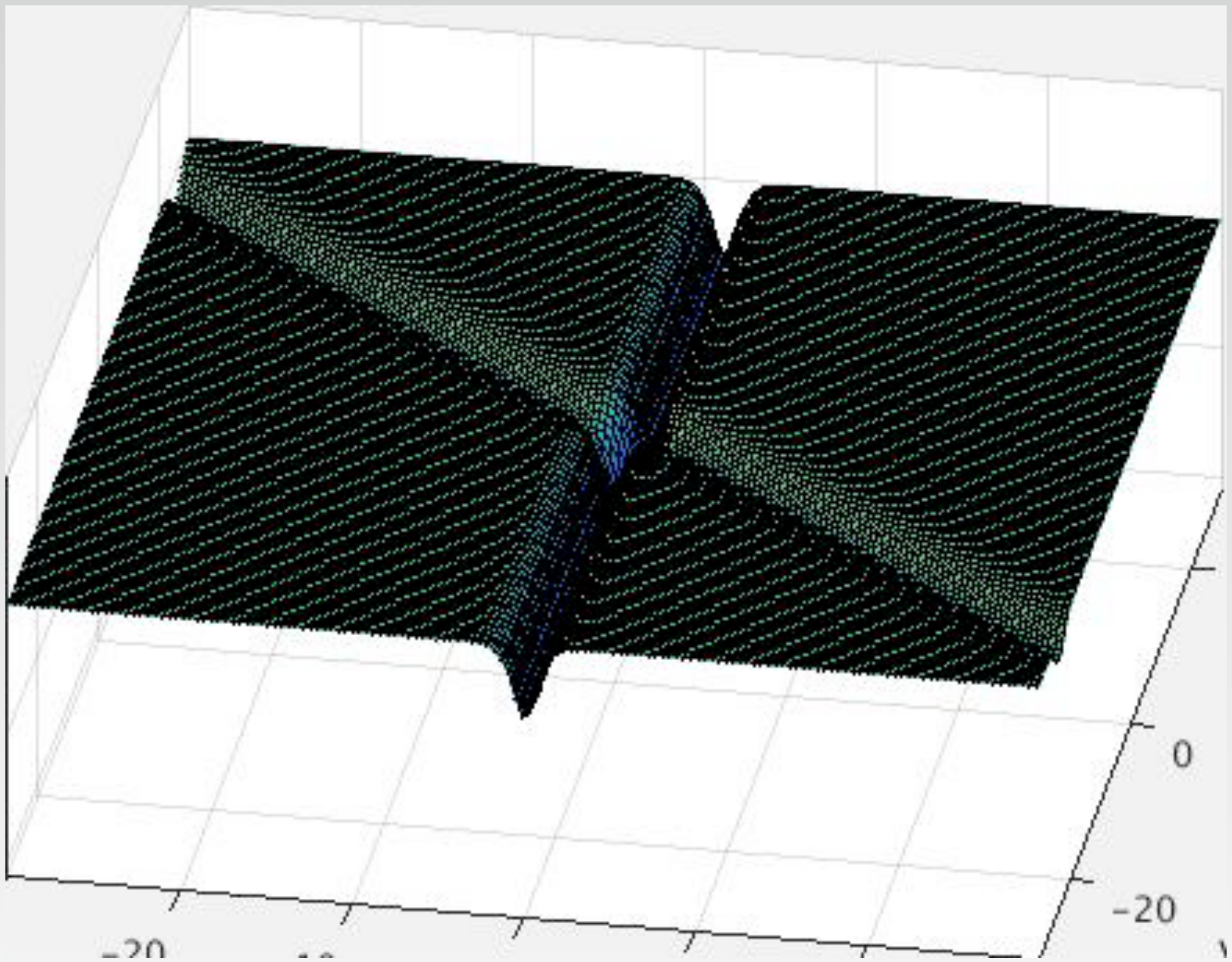
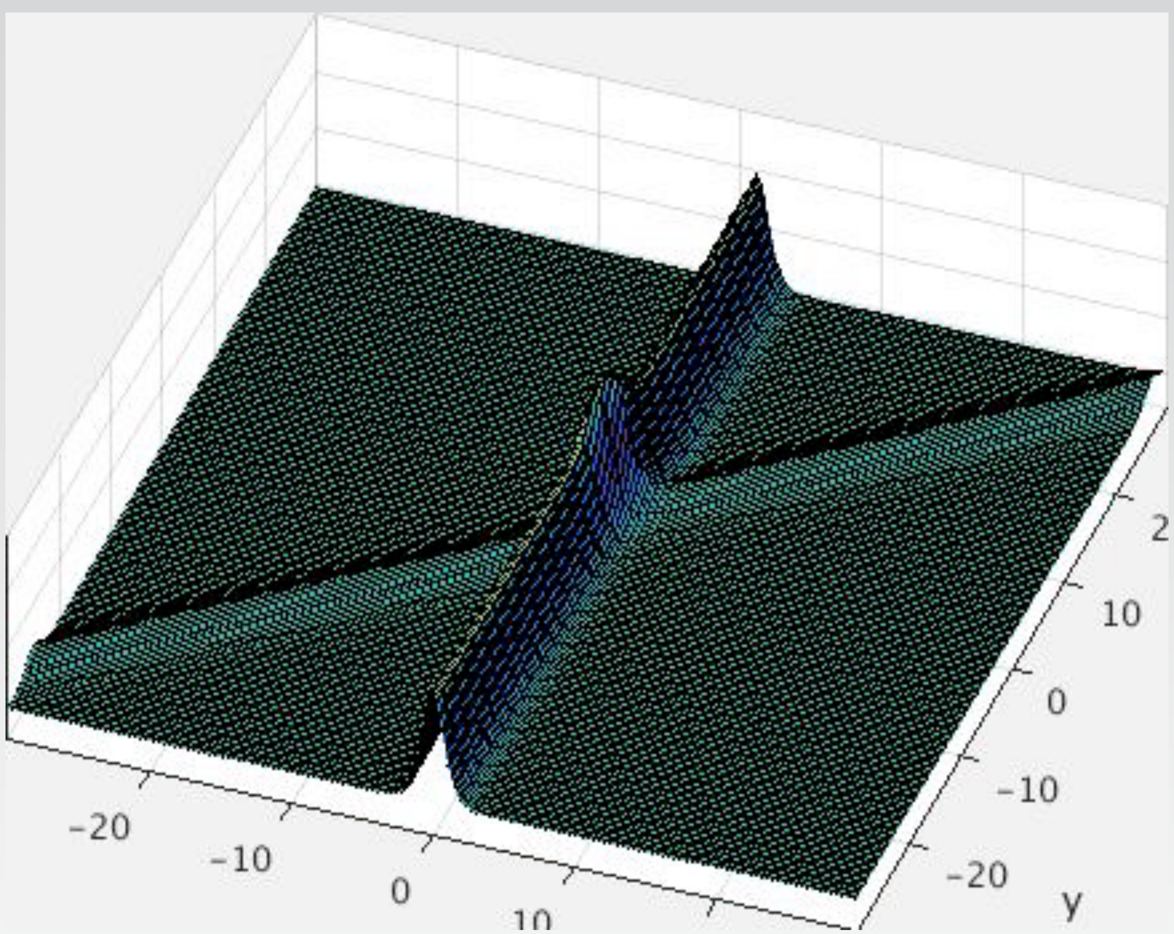


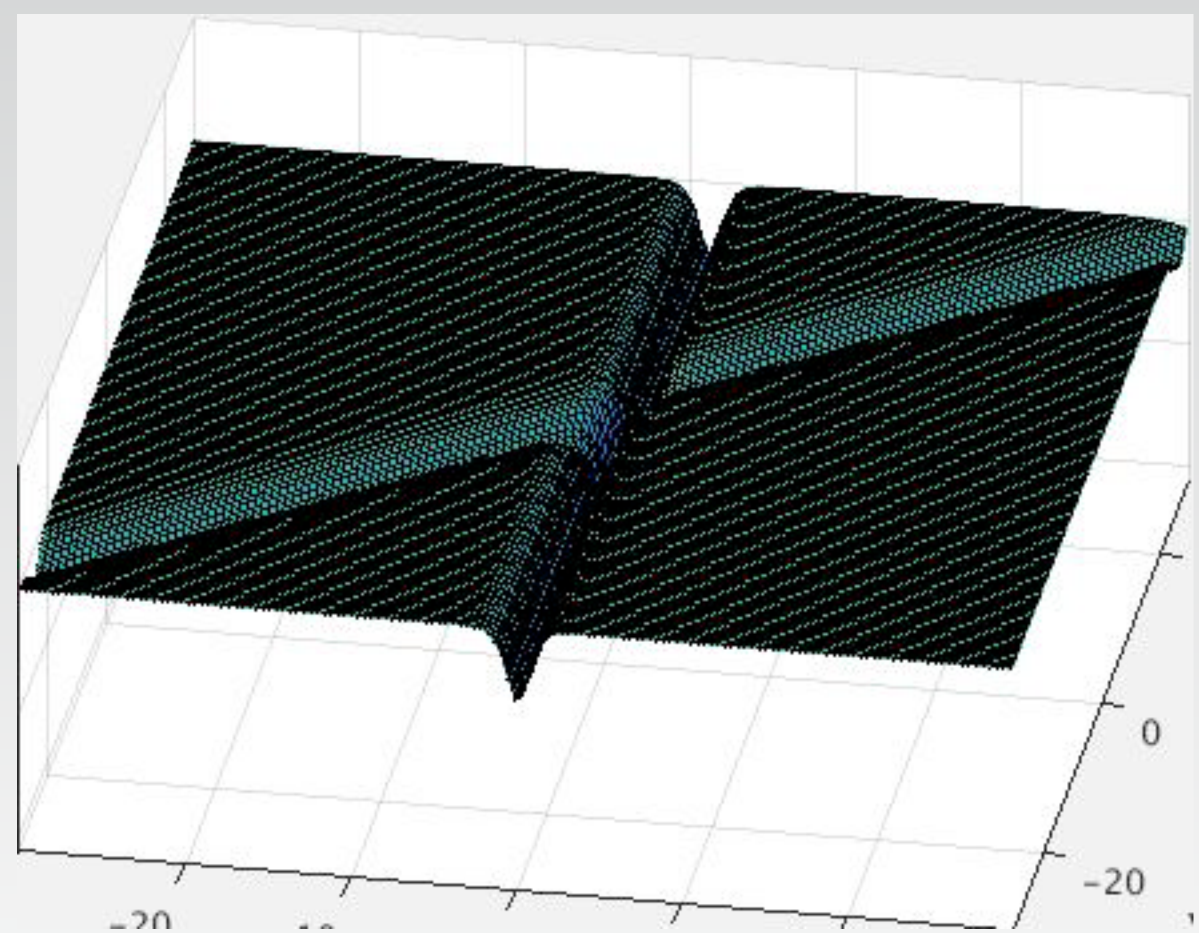
2-soliton

Angle: 130 degree $\quad A_{12} = -1.0954$

ϕ_x



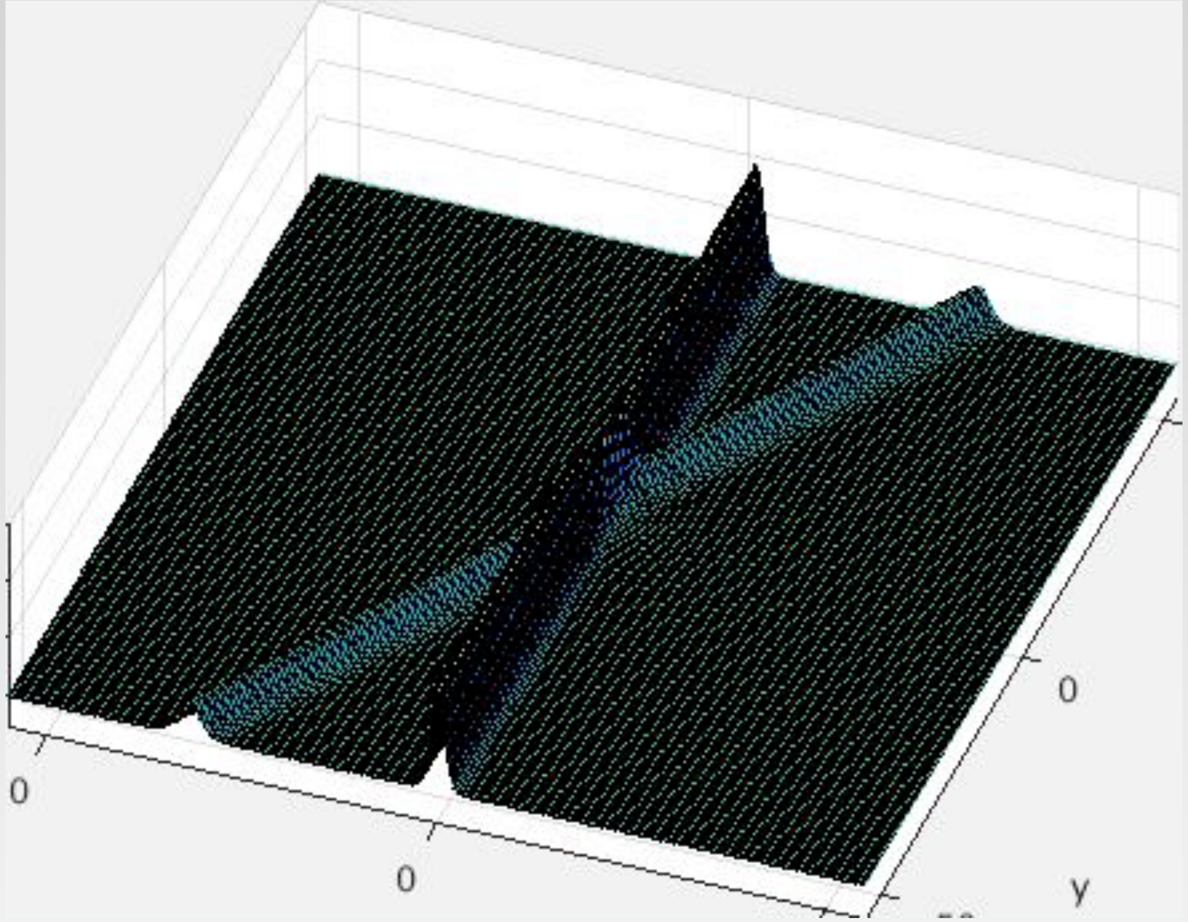
$|u|$



2-soliton

Angle: 150 degree $A_{12} = 1.5560$

ϕ_x



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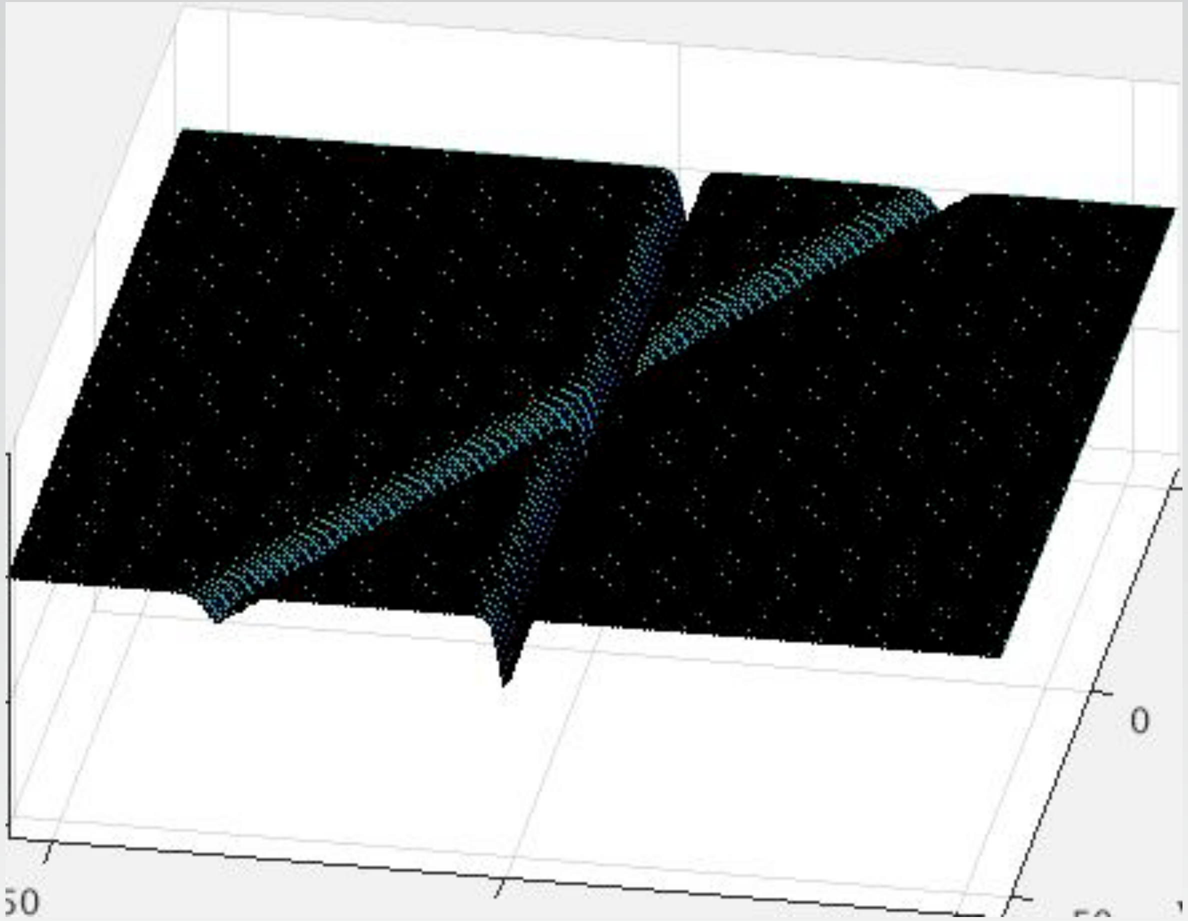


Determinant solution of DS II

Find a determinant form of N-soliton solution of DS II.

For 1-soliton solution of DS II, set

$$p = \sqrt{2}\rho_0 \cos\Psi \sin\Phi = \frac{\rho_0}{\sqrt{2}}(\sin\varphi_i - \sin\varphi_j),$$
$$q = \sqrt{2}\rho_0 \sin\Psi \sin\Phi = -\frac{\rho_0}{\sqrt{2}}(\cos\varphi_i - \cos\varphi_j),$$

$$\Psi = \frac{\varphi_i + \varphi_j}{2}, \quad \Phi = \frac{\varphi_i - \varphi_j}{2}, \quad \rho_0 = 2$$

For simplicity. We can use scale transform for other values.

φ_i, φ_j : new parameters

$$\Omega = \begin{cases} \Omega_+ = -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i + 2\sqrt{2}k\sin\varphi_j + 2\sqrt{2}l\cos\varphi_j + 2|\sin 2\varphi_i - \sin 2\varphi_j|, \\ \Omega_- = -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i + 2\sqrt{2}k\sin\varphi_j + 2\sqrt{2}l\cos\varphi_j - 2|\sin 2\varphi_i - \sin 2\varphi_j|, \end{cases}$$

Set

$$\omega_{i+} = \begin{cases} -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i + 2\sin 2\varphi_i & \text{if} \quad \sin 2\varphi_i \geq \sin 2\varphi_j \\ -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i - 2\sin 2\varphi_i & \text{if} \quad \sin 2\varphi_i < \sin 2\varphi_j \end{cases}$$

$$\omega_{i-} = \begin{cases} -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i - 2\sin 2\varphi_i & \text{if} \quad \sin 2\varphi_i \geq \sin 2\varphi_j \\ -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i + 2\sin 2\varphi_i & \text{if} \quad \sin 2\varphi_i < \sin 2\varphi_j \end{cases}$$

$$\Omega_+ = \omega_{i+} - \omega_{j+}$$
$$\Omega_- = \omega_{i-} - \omega_{j-}$$

1-soliton solution using new parameters:

$$u_{[i,j]} = 2e^{\imath(kx+ly-\omega t+\xi^{(0)})}\frac{g}{f}, \quad \phi = (\log f)_x \,, \; \theta_i = \sqrt{2}x\sin\varphi_i - \sqrt{2}y\cos\varphi_i - \omega_i t + \theta_i^{(0)}$$

$$f = a_{11}e^{\theta_i} + a_{12}e^{\theta_j} \;, g = a_{11}e^{\theta_i+\imath\varphi_i} + a_{12}e^{\theta_j+\imath\varphi_j} \;, \; g^* = a_{11}e^{\theta_i-\imath\varphi_i} + a_{12}e^{\theta_j-\imath\varphi_j},$$

$$\omega = -k^2 + l^2 + 4 \;, \quad \omega_i = -2\sqrt{2}k\sin\varphi_i - 2\sqrt{2}l\cos\varphi_i \pm 2\sin 2\varphi_i$$

$$|u_{[i,j]}|^2 = 4 - 4\sin^2\frac{\varphi_i-\varphi_j}{2}\operatorname{sech}^2\sqrt{A}(x + y\tan\Psi - Ct - x^{(0)})$$

$$\phi_x = A\operatorname{sech}^2\sqrt{A}(x + y\tan\Psi - Ct - x^{(0)})$$

$$A = \frac{(\sin\varphi_i - \sin\varphi_j)^2}{2}, \Psi = \frac{\varphi_i+\varphi_j}{2}, C = \frac{\Omega}{\sqrt{2}(\sin\varphi_i - \sin\varphi_j)}$$

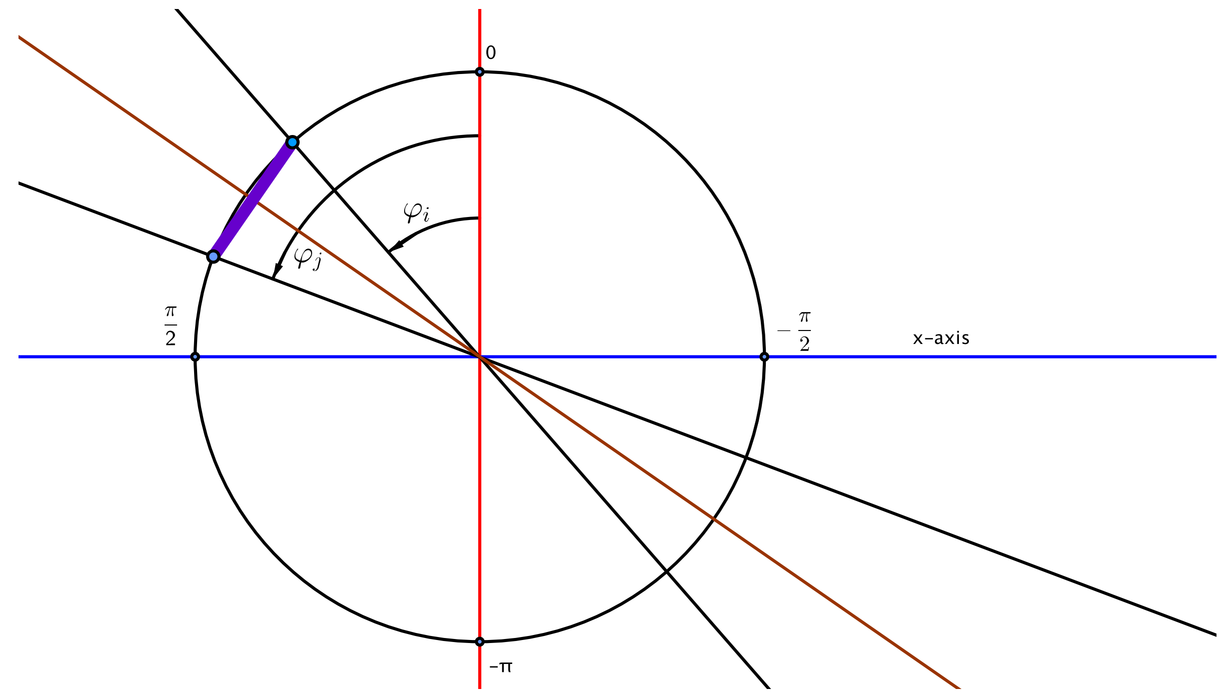
Depth of dark soliton

$$4 - 4\sin^2\frac{\varphi_i-\varphi_j}{2}$$

<2-soliton solution>

Write 2-soliton solution in determinant form: ($-\pi \le \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 < \pi$)

$$f = \tau^{(-\frac{1}{2})} = \begin{vmatrix} \psi_1^{(-\frac{1}{2})} & \psi_1^{(\frac{1}{2})} \\ \psi_2^{(-\frac{1}{2})} & \psi_2^{(\frac{1}{2})} \end{vmatrix}$$



$$g = \tau^{(\frac{1}{2})} = \begin{vmatrix} \psi_1^{(\frac{1}{2})} & \psi_1^{(\frac{3}{2})} \\ \psi_2^{(\frac{1}{2})} & \psi_2^{(\frac{3}{2})} \end{vmatrix}$$

$$g^* = \tau^{(-\frac{3}{2})} = \begin{vmatrix} \psi_1^{(-\frac{3}{2})} & \psi_1^{(-\frac{1}{2})} \\ \psi_2^{(-\frac{3}{2})} & \psi_2^{(-\frac{1}{2})} \end{vmatrix}$$

$$\psi_i^{(n)} = \sum_{j=1}^{4} a_{ij} e^{\sqrt{2}x \sin\varphi_j - \sqrt{2}y \cos\varphi_j - \omega_j t + n\imath\varphi_j}$$

$$\tau^{(n)} = |\psi_i^{(n+j-1)}|_{1 \le i,j \le 2}$$

$$\omega = -k^2 + l^2 + 4 \;, \qquad \omega_i = -2\sqrt{2}k \sin\varphi_i - 2\sqrt{2}l \cos\varphi_i \pm 2\sin 2\varphi_i$$

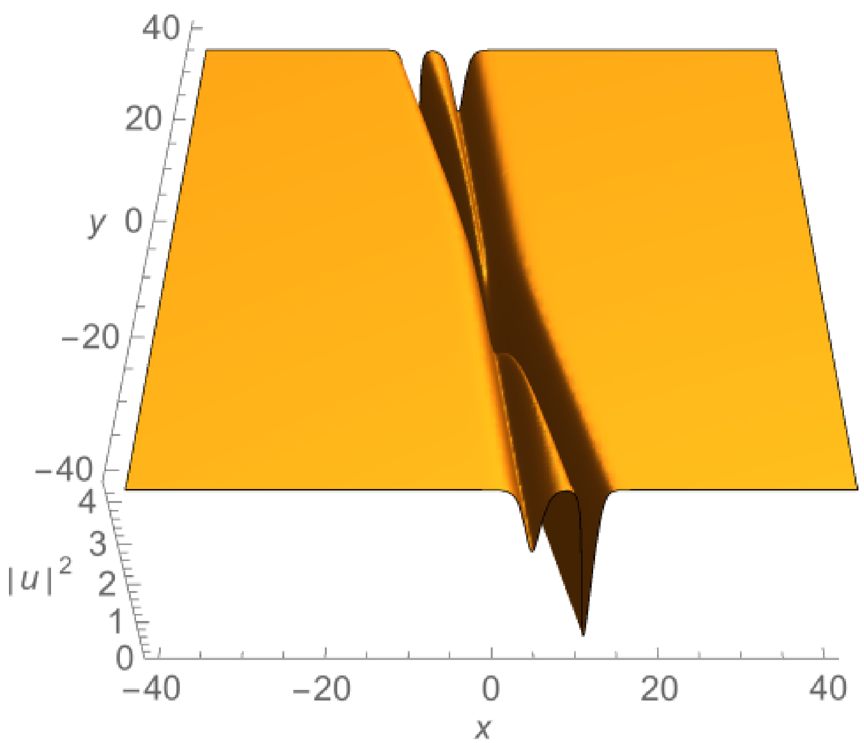
This can be obtained by the redcution of KP Wronskian solution.

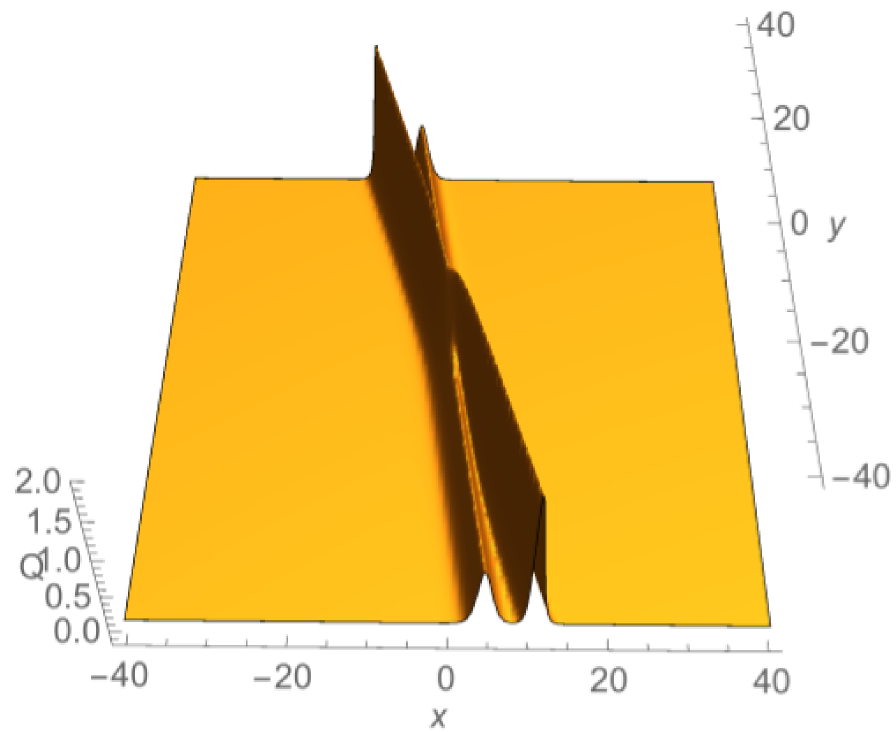
Case1<P-type>

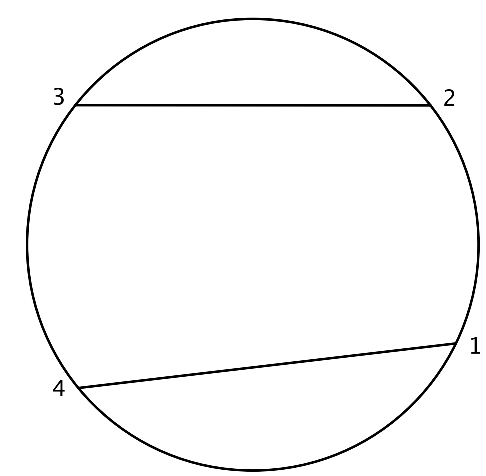
Use P-type A matrix for KP:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix}, \quad a, b > 0$$

$$A_{12} = \frac{\sin\frac{\varphi_2-\varphi_1}{2}\sin\frac{\varphi_4-\varphi_3}{2}}{\sin\frac{\varphi_3-\varphi_1}{2}\sin\frac{\varphi_4-\varphi_2}{2}}$$





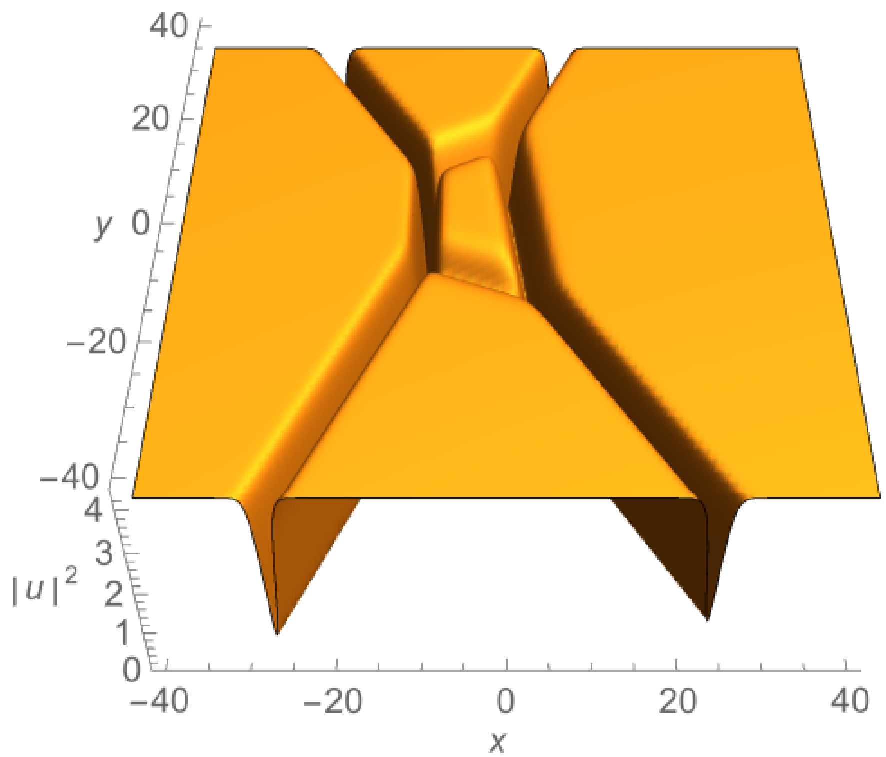


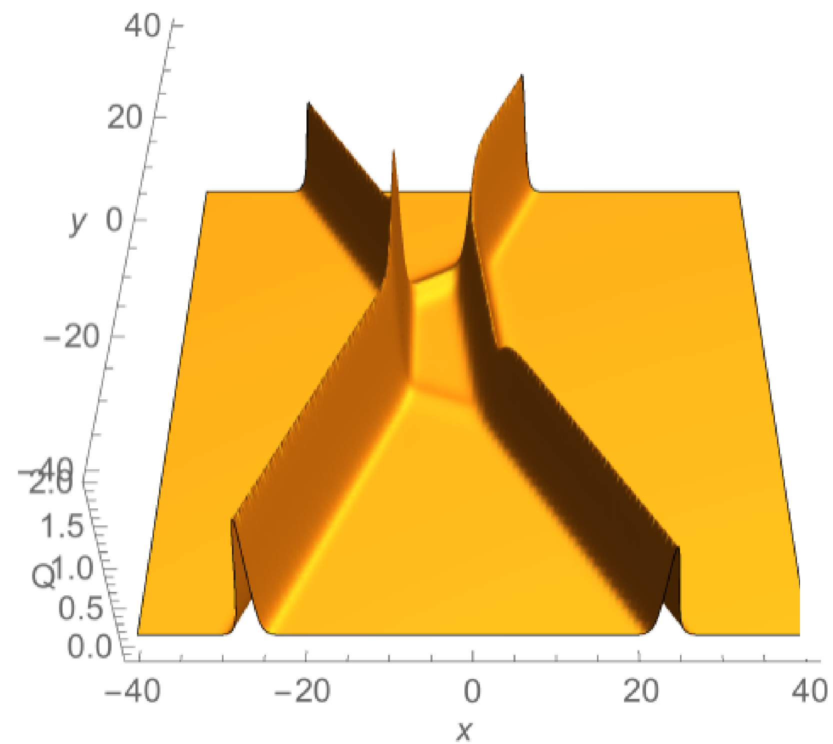
chord diagram

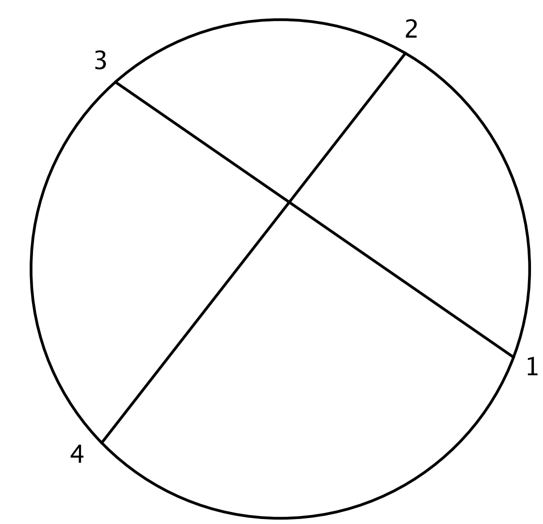
Case2 <T-type>

Use T-type A matrix for KP:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix} \quad a, b, c, d > 0, \quad ad - bc > 0,$$







chord diagram

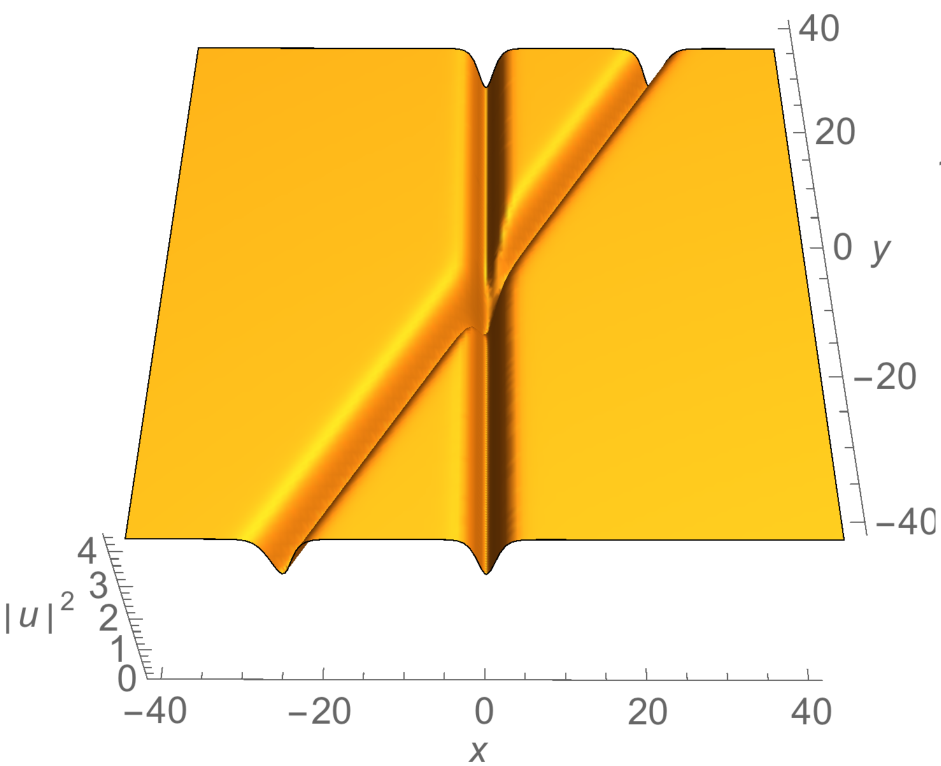
Remark： There is a region in which soliton interaction is similar to P-type, we call this P2-type.

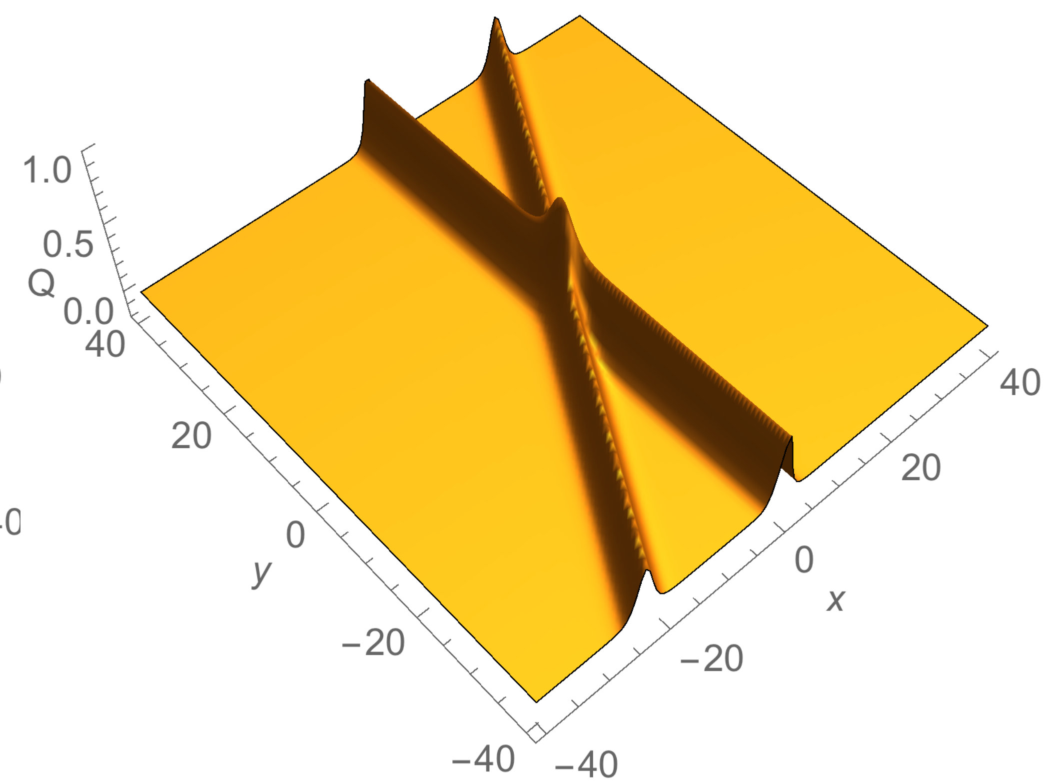
Case3<O-type>

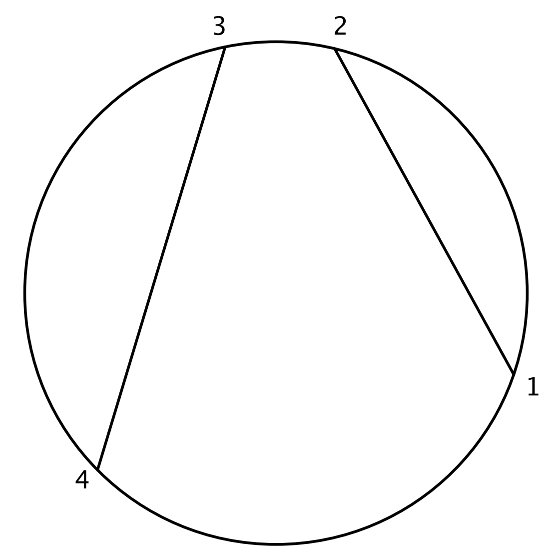
Use O-type A-matrix for KP:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \quad a, b > 0,$$

$$A_{12} = \frac{\sin\frac{\varphi_3-\varphi_1}{2}\sin\frac{\varphi_4-\varphi_2}{2}}{\sin\frac{\varphi_3-\varphi_2}{2}\sin\frac{\varphi_4-\varphi_1}{2}}$$



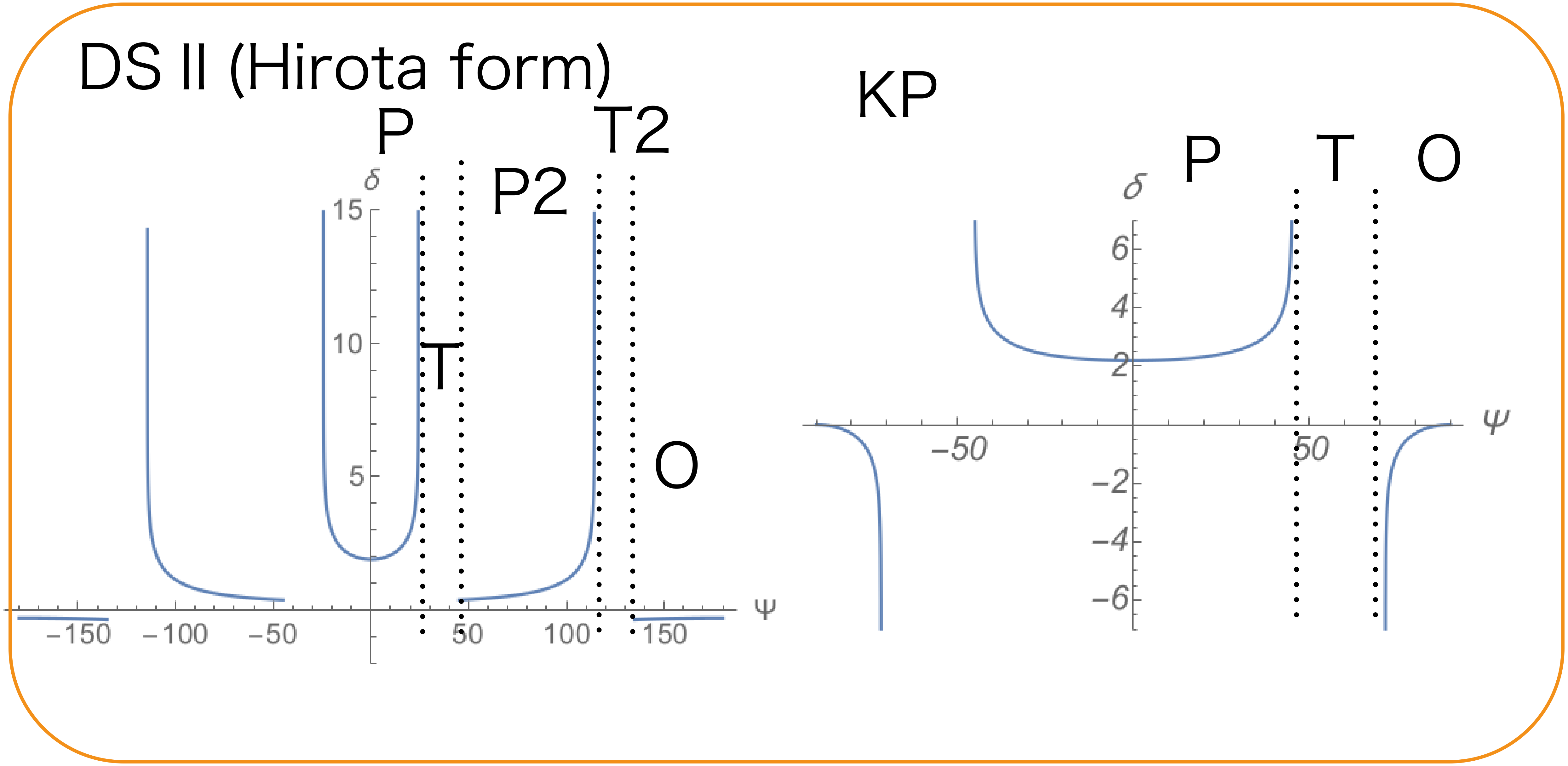




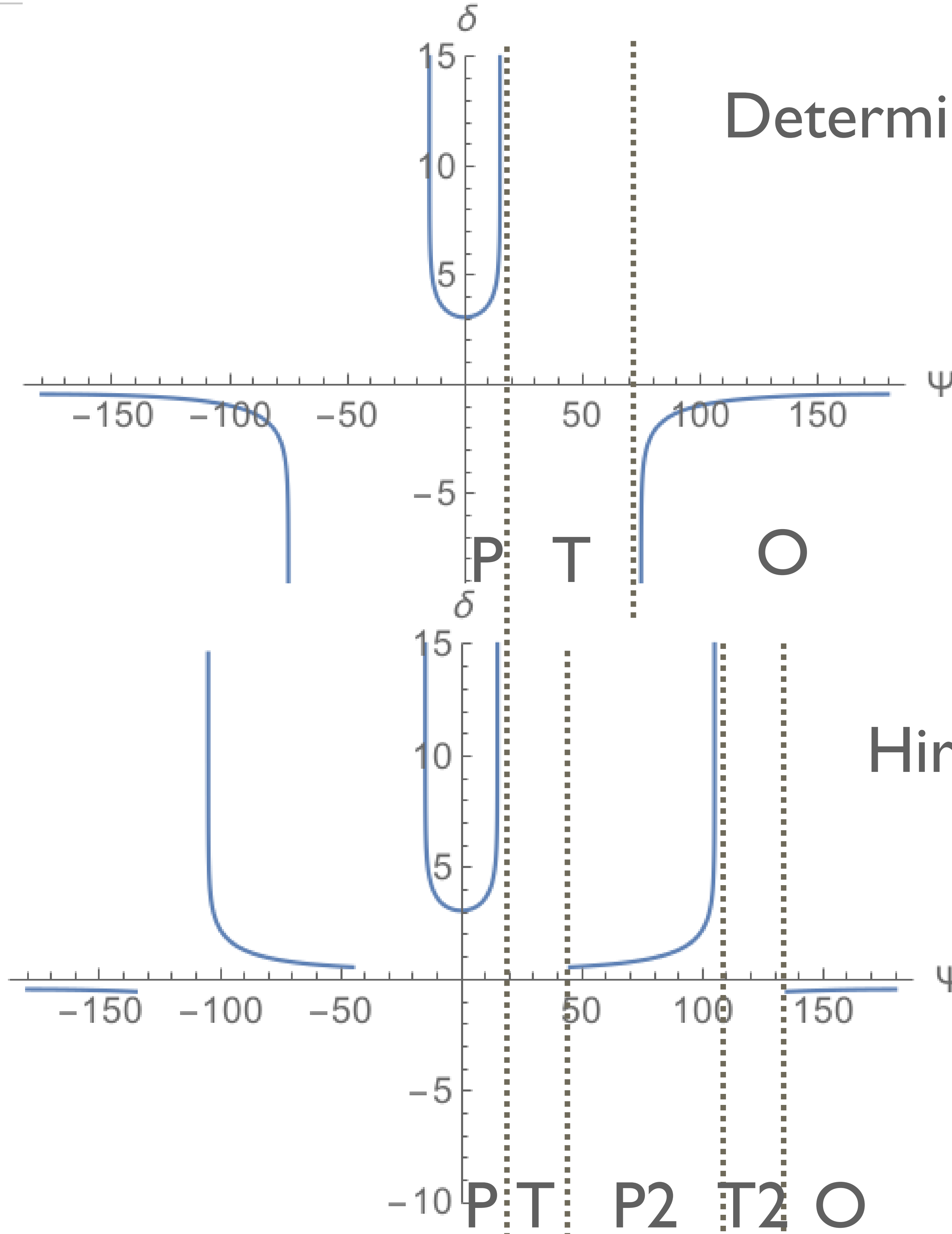
chord diagram

Remark： There is a region in which soliton interaction is similar to P-type, we call this P2-type.
There is a region in which soliton interaction is similar to T-type, we call this T2-type.

Angle dependency of 2-soliton solution



$p_1 = 2, q_1 = 0, p_2 = \cos\Psi, q_2 = \sin\Psi$



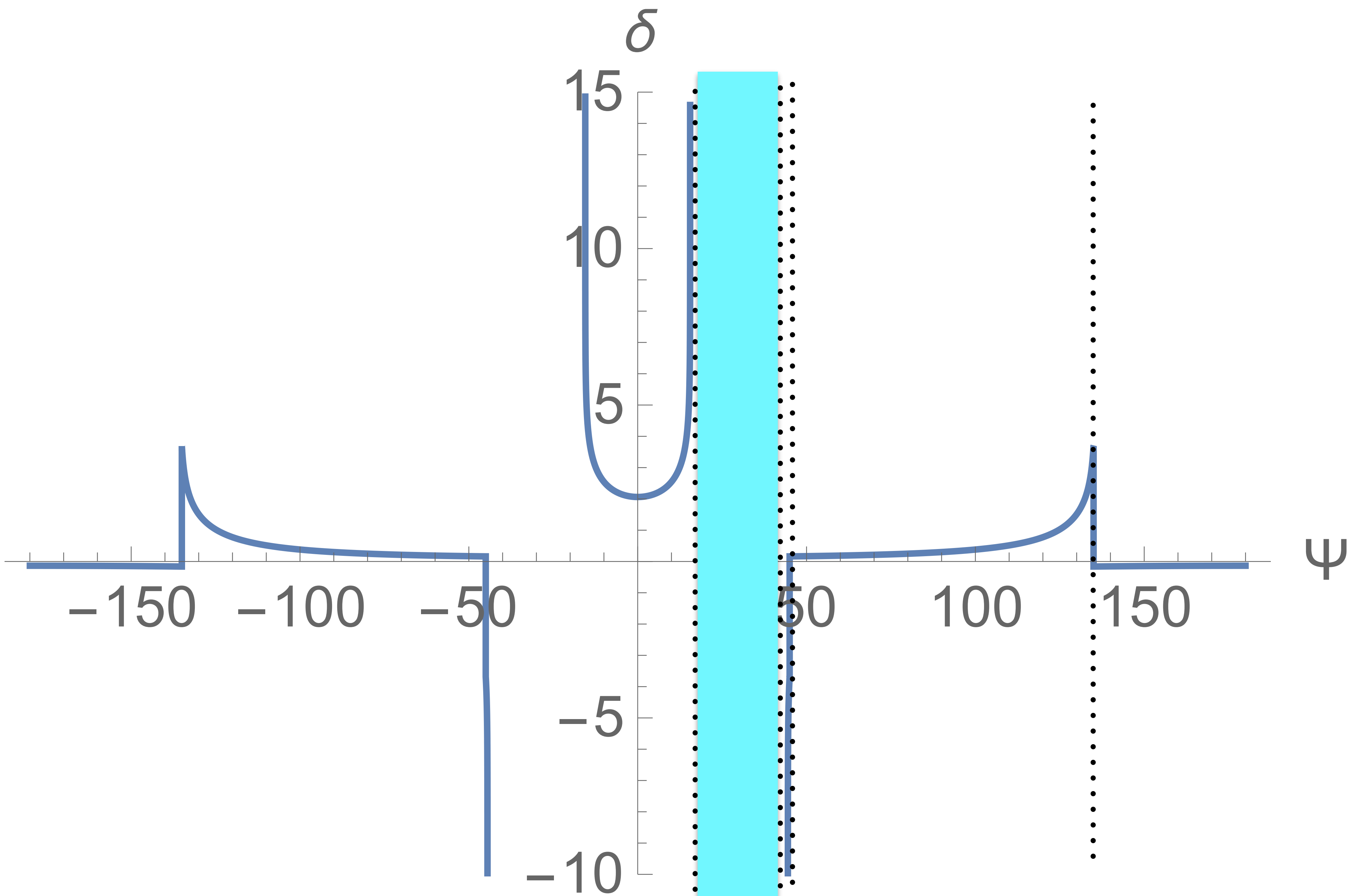
Determinant solution of DS II

Hirota form for DS II

Numerial experiments

Angle dependency of DS II 2-soliton solution

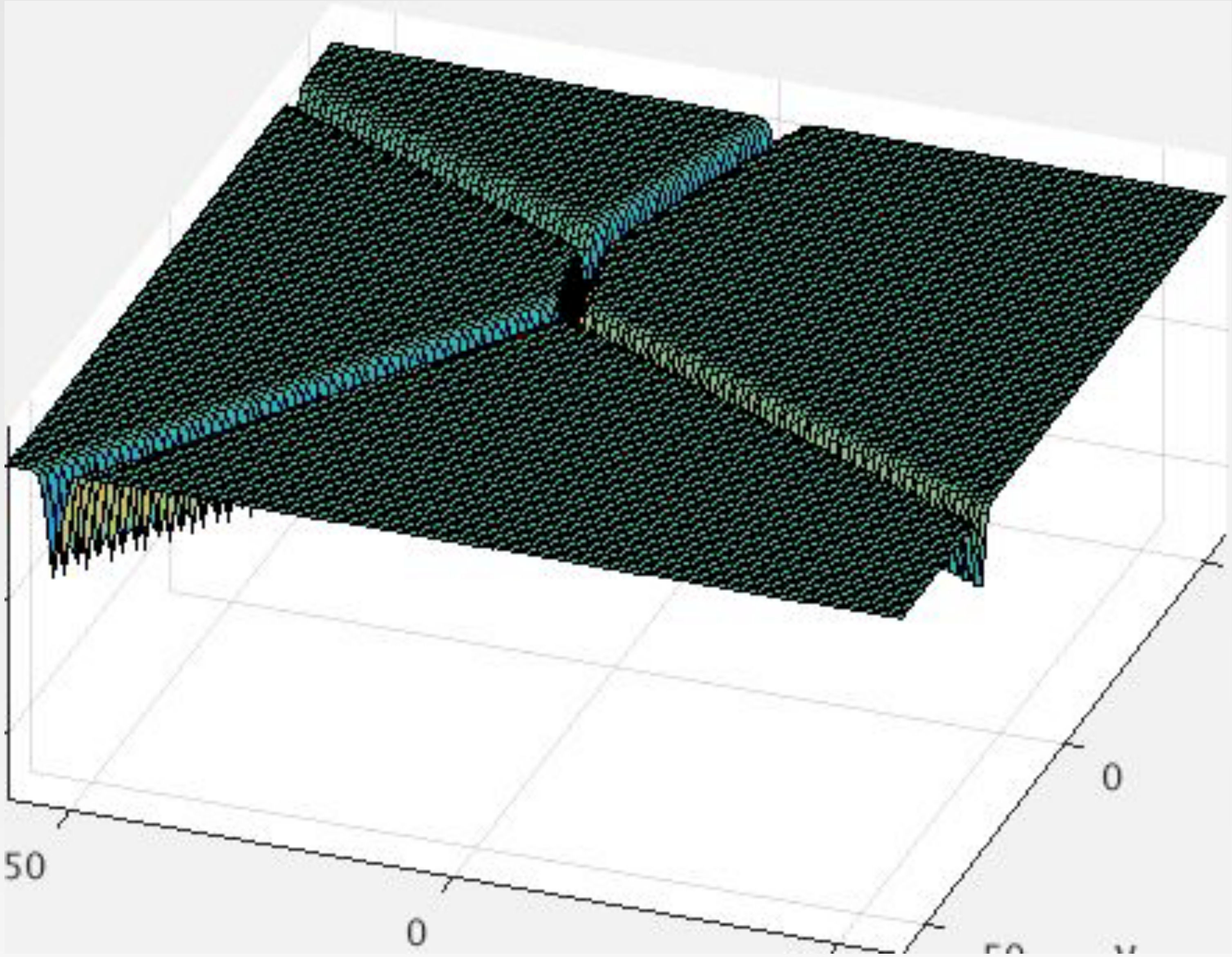
for small depth



Soliton Reconnection

(Theory : Nishinari, Abe, Satuma 1992)

$$(\psi_1 = -0.51\pi, \psi_2 = -0.2\pi, \psi_3 = 0, 4\pi, \psi_4 = 0.9\pi, \rho_0 = 2)$$



2 soliton resonances

$$f = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4}$$

$$g = e^{\theta_1 + i\psi_1} + e^{\theta_2 + i\psi_2} + e^{\theta_3 + i\psi_3} + e^{\theta_4 + i\psi_4}$$

$$\theta_i = \sqrt{2}x \sin\psi_i - \sqrt{2}y \cos\psi_i - (-2\sqrt{2}k \sin\psi_i - 2\sqrt{2}l \cos\psi_i + 2\sin 2\psi_i)t + \theta_i^0$$

Summary

- Developed a numerical method for DS II dark line solitons,
- Investigated dark line soliton interactions of DS II theoretically and numerically.
- Future problem：Develop a theory which explains all transitions among different line soliton interactions of DS II by using circular chord diagrams.

PART 2: SELF-ADAPTIVE MOVING MESH SCHEMES

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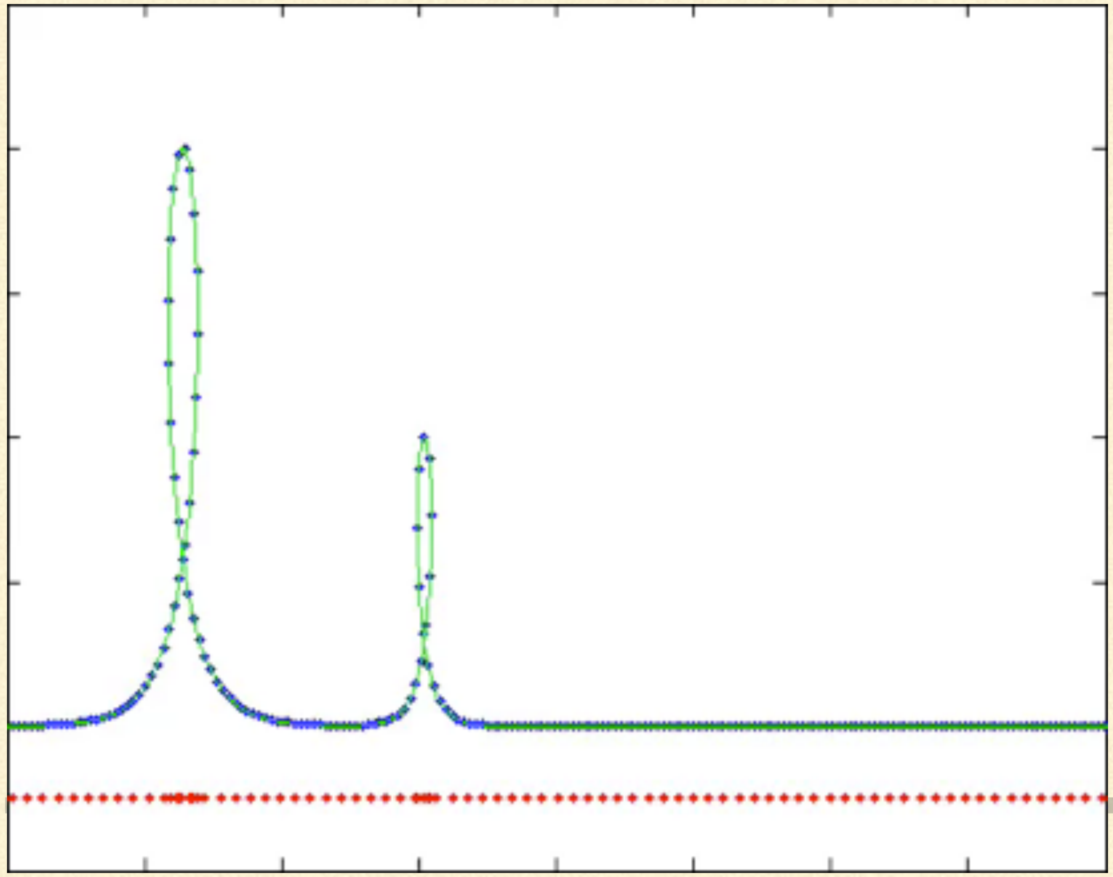
Ayaka Hata (student)

SELF-ADAPTIVE MOVING MESH SCHEME”
(FENG-KM-OHTA)

- A novel numerical difference method for solving nonlinear PDEs.
- This new method was obtained from integrable discretization of integrable PDEs.
- Mesh points are generated automatically.
- Mesh is refined around regions of large deformation.

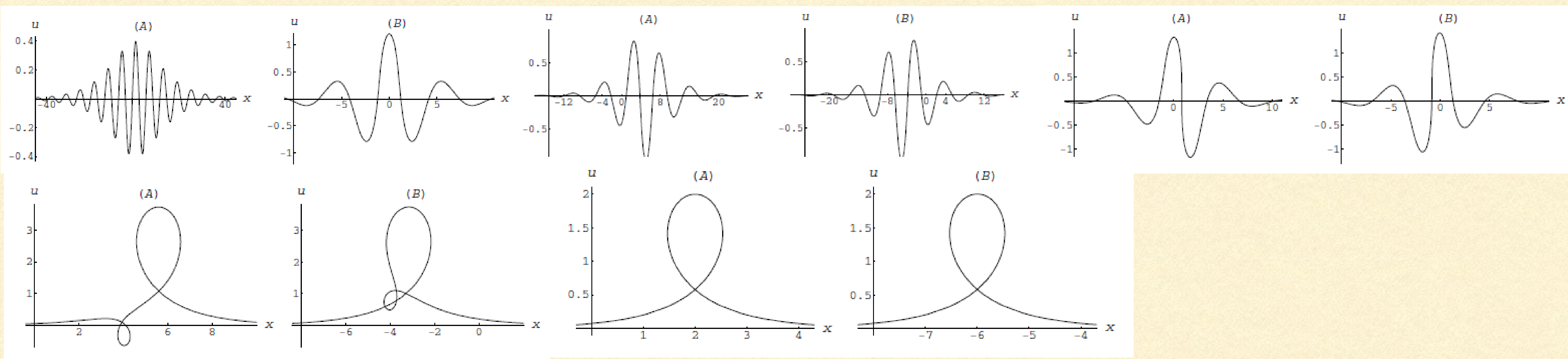


Problem

How to construct self-adaptive moving mesh schemes in 3 dimensions

SHORT PULSE EQUATION

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$



Plots of Exact Solutions (different parameters)

HODOGRAPH TRANSFORMATION

Integrable Coupled dispersionless system (Konno, Kakuhata)

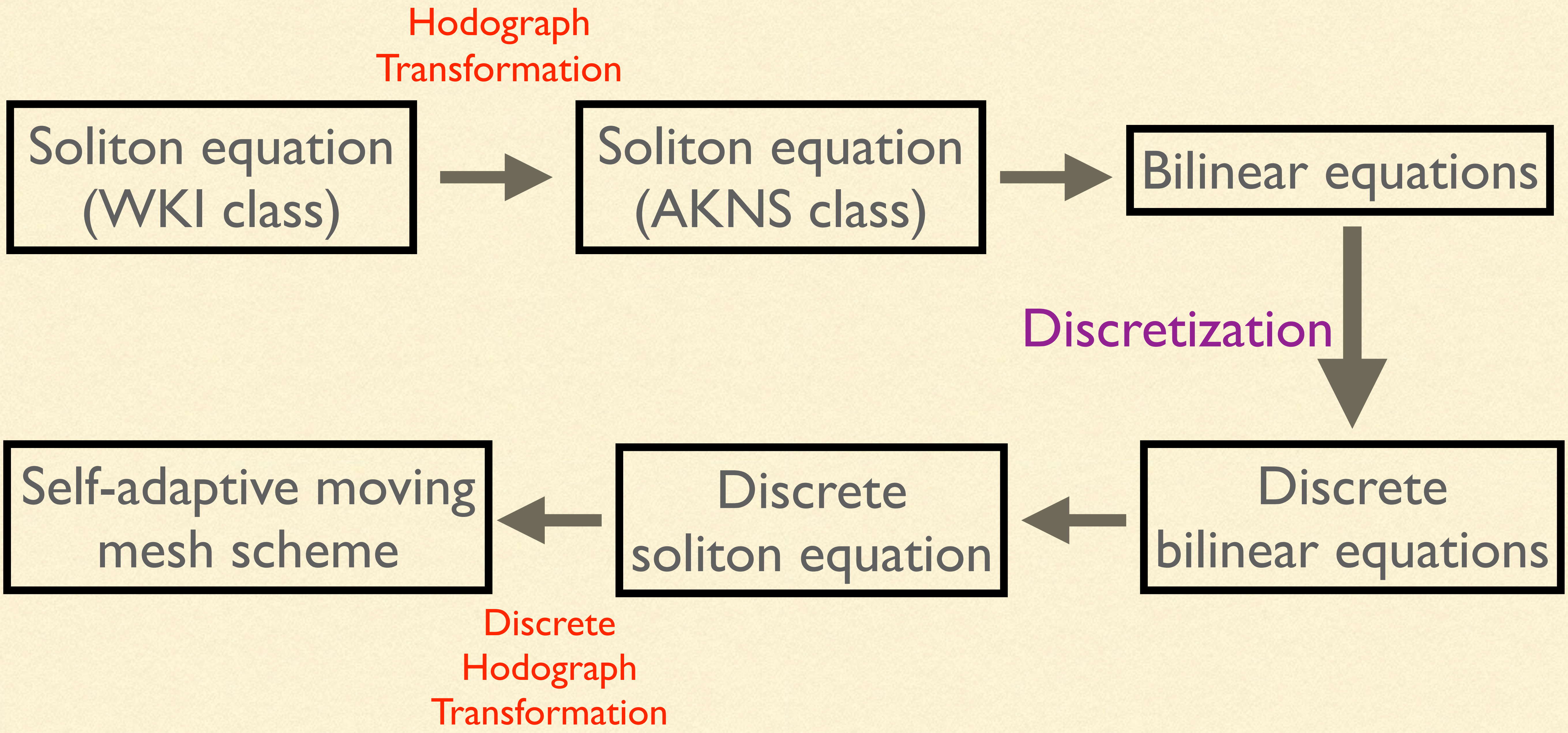
$$\frac{\partial \rho}{\partial T} + \frac{\partial}{\partial X}\left(\frac{u^2}{2}\right) = 0$$ ⟵ Conservation Law

$$\frac{\partial^2 u}{\partial X \partial T} = \rho u$$

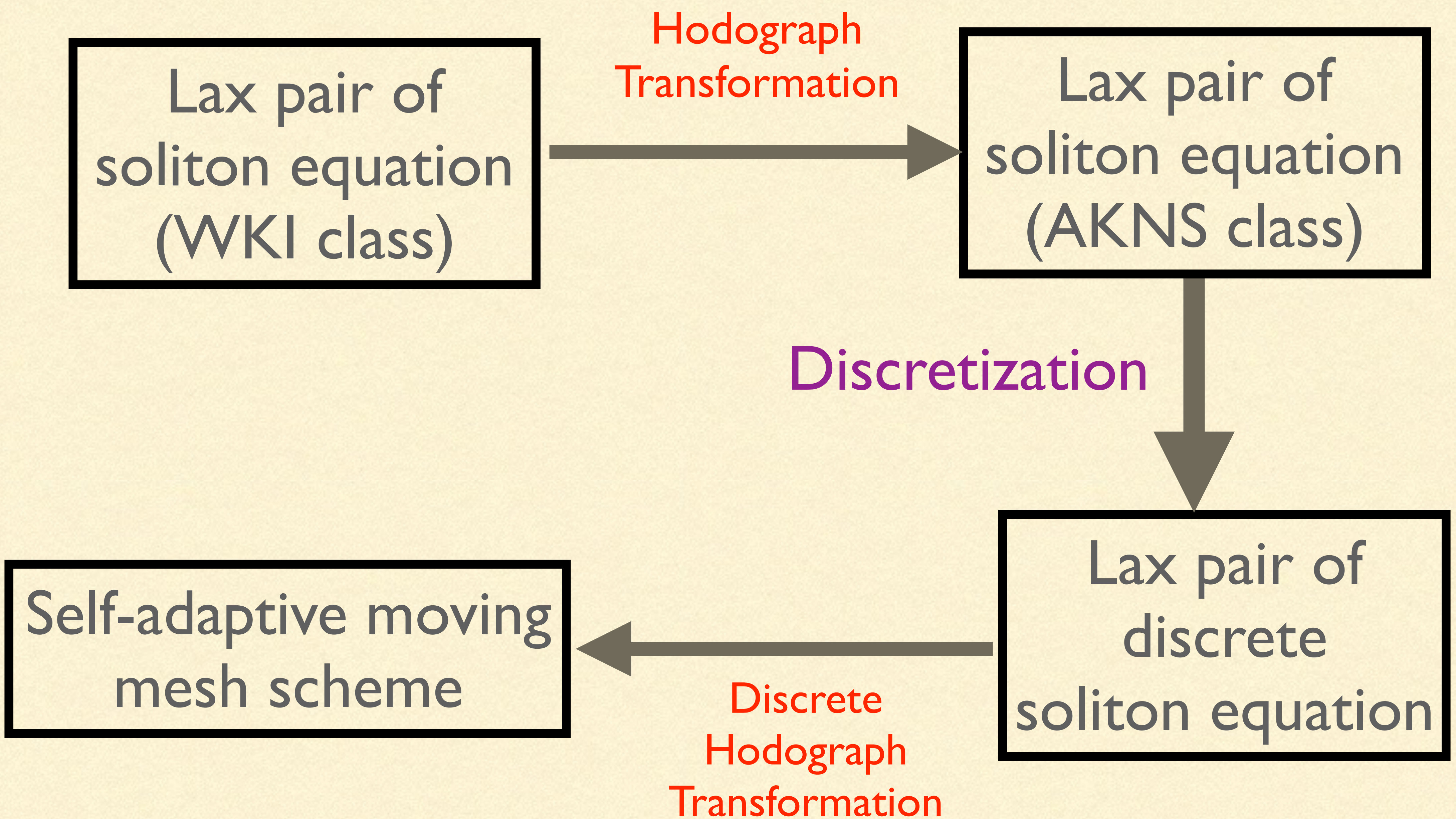
Hodograph transformation

$$x = \int \rho(X,T)\,dX\,, \quad t = T\,.$$

DISCRETIZATION METHOD
(USE BILINEAR EQUATIONS)



DISCRETIZATION METHOD
(USE LAX PAIRS)



DISCRETIZATION BY LAX PAIRS

Step 1

Consider the linear 2×2 system (Lax pair) for the SP equation:

$$\frac{\partial \Psi}{\partial x} = \tilde{U}\Psi\,, \qquad \frac{\partial \Psi}{\partial t} = \tilde{V}\Psi\,,$$

$$\tilde{U} = -\mathrm{i}\lambda \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} - \frac{\mathrm{i}\lambda}{2}u^2 & -\frac{u}{2} - \frac{\mathrm{i}\lambda}{2}u^2 u_x \\ \frac{u}{2} - \frac{\mathrm{i}\lambda}{2}u^2 u_x & -\frac{\mathrm{i}}{4\lambda} + \frac{\mathrm{i}\lambda}{2}u^2 \end{pmatrix},$$

hodograph transformation $\Downarrow$ $x = X_0 + \int_{X_0}^{X} \rho(\tilde{X}, T) d\tilde{X}\,, \quad t = T$

$$\frac{\partial \Psi}{\partial X} = U\Psi\,, \qquad \frac{\partial \Psi}{\partial T} = V\Psi\,,$$

$$U = -\mathrm{i}\lambda \begin{pmatrix} \rho & u_X \\ u_X & -\rho \end{pmatrix}, \quad V = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} & -\frac{u}{2} \\ \frac{u}{2} & -\frac{\mathrm{i}}{4\lambda} \end{pmatrix}.$$

DISCRETIZATION BY LAX PAIRS

Step 2:

Discretize the linear system obtained in Step 2:

$$\Psi_{k+1} = U_k \Psi_k \,, \qquad \frac{\partial \Psi_k}{\partial T} = V_k \Psi_k \,,$$

$$U_k = \begin{pmatrix} 1 - \mathrm{i}\lambda a \rho_k & -\mathrm{i}\lambda \frac{u_{k+1} - u_k}{a} \\ -\mathrm{i}\lambda \frac{u_{k+1} - u_k}{a} & 1 + \mathrm{i}\lambda a \rho_k \end{pmatrix} ,$$

$$V_k = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} & -\frac{u_k}{2} \\ \frac{u_k}{2} & -\frac{\mathrm{i}}{4\lambda} \end{pmatrix} ,$$

DISCRETIZATION BY LAX PAIRS

Step 3:

Discretize the hodograph transformation $x = X_0 + \int_{X_0}^{X} \rho(\tilde{X}, T) d\tilde{X}$:

$$x_k = X_0 + \sum_{j=0}^{k-1} a\rho_j \, .$$

x_k is a lattice point, a mesh interval is defined as $\delta_k = x_{k+1} - x_k$. By the discrete hodograph transformation, δ and ρ satisfy the relation $\delta_k = a\rho_k$. Using this relation, the Lax pair is rewritten as

$$U_k = \begin{pmatrix} 1 + \lambda\delta_k & \lambda\frac{u_{k+1} - u_k}{a} \\ \lambda\frac{u_{k+1} - u_k}{a} & 1 - \lambda\delta_k \end{pmatrix},$$

$$V_k = \begin{pmatrix} \frac{1}{4\lambda} & -\frac{u_k}{2} \\ \frac{u_k}{2} & -\frac{1}{4\lambda} \end{pmatrix}.$$

DISCRETIZATION BY LAX PAIRS

Step 4:

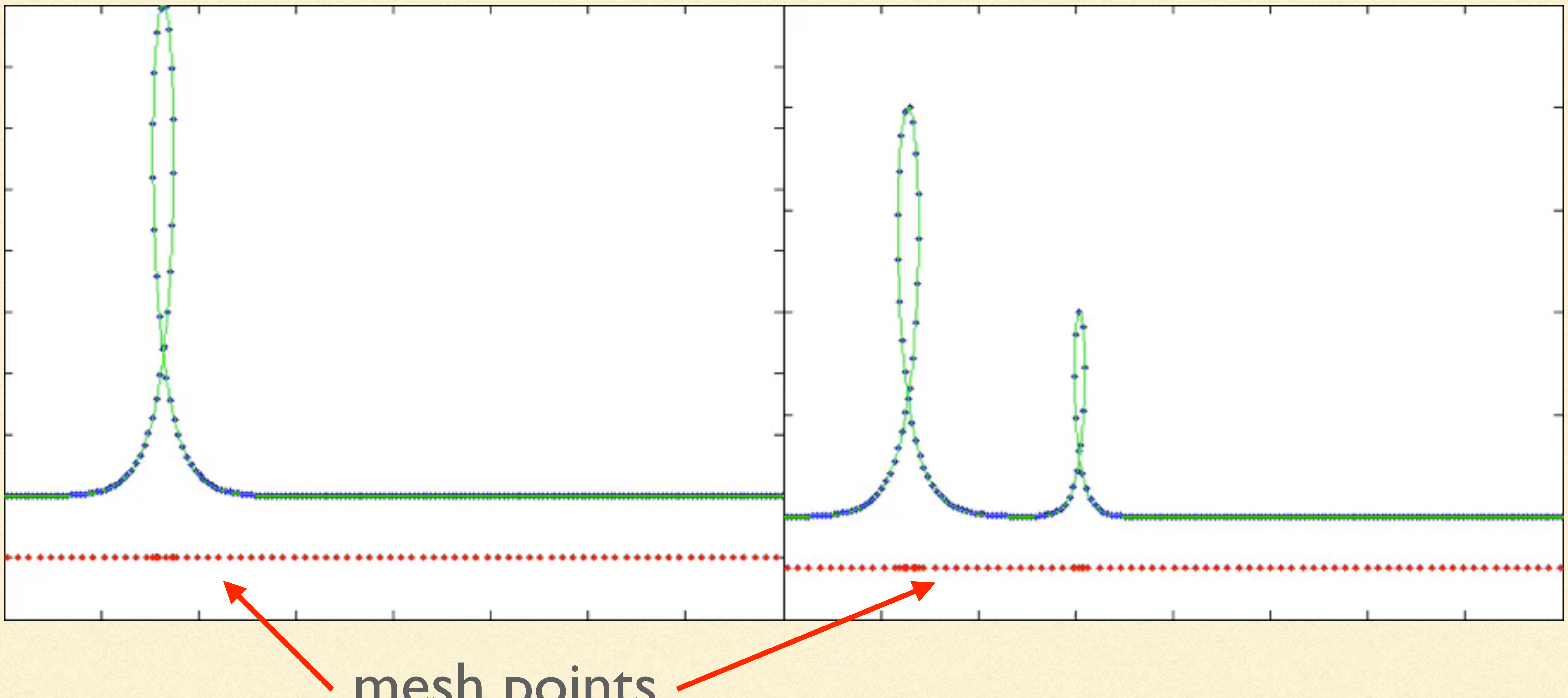
The compatibility condition gives

$$\partial_T \delta_k = \frac{-u_{k+1}^2 + u_k^2}{2},$$

$$\partial_T (u_{k+1} - u_k) = \delta_k \frac{u_{k+1} + u_k}{2}.$$

With the hodograph transformation $\delta_k = x_{k+1} - x_k$ ($x_k = X_0 + \sum_{j=0}^{k-1} \delta_j$), this system gives a semi-discrete analogue of the SP equation. The set $\{x_k, u_k\}$ gives solutions.

NUMERICAL SIMULATION (FMO2014)



Time evolution ➡ improved Euler method

WHY DOES A SELF-ADAPTIVE MOVING MESH SCHEME WORK WELL?

$$\partial_T \delta_k = \frac{-u_{k+1}^2 + u_k^2}{2} \, ,$$

$$\partial_T (u_{k+1} - u_k) = \delta_k \frac{u_{k+1} + u_k}{2} \, ,$$

Discrete conservation law

$\delta_k \equiv a\rho_k = x_{k+1} - x_k$ is a conserved density.

If $u_{k+1}^2 > u_k^2$, δ_k decreases. → Mesh is refined.

If $u_{k+1}^2 < u_k^2$, δ_k increases. → Mesh is de-refined.

SELF-ADAPTIVE MESH SCHEME OF HUNTER-SAXTON EQUATION (FENG-OHTA-KM 2010)

$$w_{txx} - 2\kappa^2 w_x + 2w_x w_{xx} + w w_{xxx} = 0$$

$$\frac{1}{2}(\ln\rho)_{XT} = \frac{\rho}{2} - \frac{2}{\rho}\,,$$ sinh-Gordon equation

$$\rho_T = w_X$$

$$x = \int w(X,T)dT = X_0 + \int^X \rho(\tilde{X},T)d\tilde{X}\,, \quad t = T$$

Hodograph transformation

$$(\partial_t + w\partial_x)(1 - w_{xx}) = -2w_x(1 - w_{xx})$$ Hunter-Saxton eq.

Discretization

$$\left(\frac{w_{k+1}-w_k}{\delta_k} - \frac{w_k - w_{k-1}}{\delta_{k-1}}\right) = \frac{\delta_k}{2} - \frac{2a^2}{\delta_k} + \frac{\delta_{k-1}}{2} - \frac{2a^2}{\delta_{k-1}}\,,$$

$$\partial_T \delta_k = w_{k+1} - w_k\,,$$

$$\delta_k \equiv a\rho_k = x_{k+1} - x_k$$

$$x_k = \int w_k dT = X_0 + \sum_{j=0}^{k-1} a\rho_j\,, \quad t = T$$

SELF-ADAPTIVE MESH SCHEME OF CAMASSA-HOLM EQUATION

$$w_t + 2\kappa^2 w_x - w_{txx} + 3ww_x = 2w_x w_{xx} + ww_{xxx}$$

$$\frac{1}{2}(\ln\rho)_{XT} = \frac{\rho}{2c} - \frac{2c}{\rho} + \frac{\rho}{2}w$$

deformed sinh-Gordon equation

(1-parameter deformation)

$$\rho_T = w_X$$

$$x = \int w(X,T)dT = X_0 + \int^X \rho(\tilde{X},T)d\tilde{X}\,, \quad t = T$$

Hodograph transformation

$$(\partial_t + w\partial_x)\left(\frac{1}{c} + w - w_{xx}\right) = -2w_x\left(\frac{1}{c} + w - w_{xx}\right)$$

CH eq.

Discretization

$$\partial_{x_{-1}}\rho_k = \frac{w_{k+1} - w_k}{a}\,,$$

$$\delta_k = 2\frac{(1+ac)e^{a(\rho_k - 2c)} - (1-ac)}{(1+ac)e^{a(\rho_k - 2c)} + (1-ac)}\,,$$

$$\frac{2}{\delta_k}(w_{k+1} - w_k) - \frac{2}{\delta_{k-1}}(w_k - w_{k-1}) = \frac{\delta_k}{2}(w_{k+1} + w_k) + \frac{\delta_k}{c}\left(1 - \frac{4a^2c^2}{\delta_k^2}\right) + \frac{\delta_{k-1}}{2}(w_k + w_{k-1}) + \frac{\delta_{k-1}}{c}\left(1 - \frac{4a^2c^2}{\delta_{k-1}^2}\right)$$

$$\delta_k \equiv a\rho_k = x_{k+1} - x_k$$

$$x_k = \int w_k dT = X_0 + \sum_{j=0}^{k-1} a\rho_j\,, \quad t = T$$

SELF-ADAPTIVE MOVING MESH SCHEMES FROM A GEOMETRIC POINT OF VIEW

- Self-adaptive moving mesh schemes can be interpreted as equations describing a motion of discrete curves.
- Equations in the mKdV hierarchy describe a motion of a continuous curve, and the space variable corresponds to an arc-length parameter and the dependent variable corresponds to curvature. Lagrangian description
- Equations in the WKI hierarchy (e.g. short pulse equation) describe a motion of a continuous curve in Cartesian coordinates. Eulerian description

SOLITON EQUATIONS AND MOTION OF A CURVE

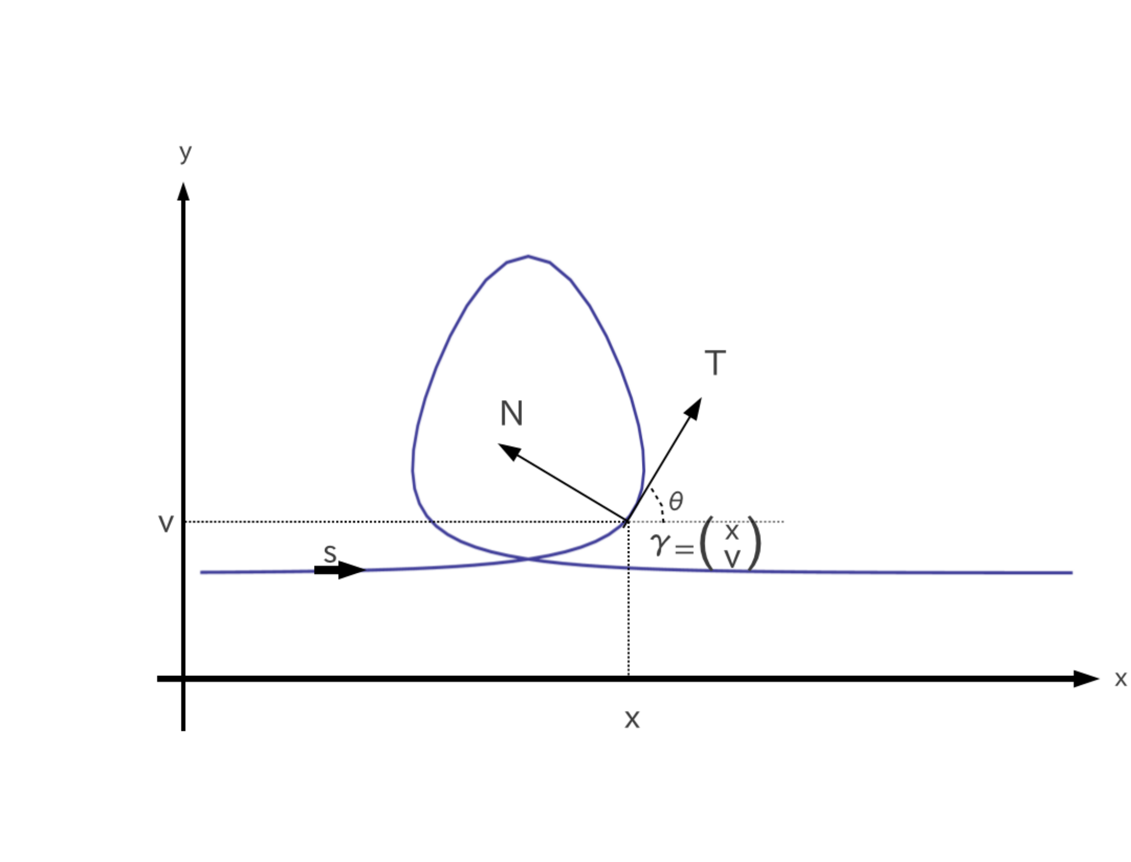
Goldstein& Petrich 1991

A curve $\gamma(s)$, s: arc length parameter.

Tangent vector $\mathbf{T} = \frac{\partial \gamma}{\partial s} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $|\mathbf{T}| = 1$.

Normal vector $\mathbf{N} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

$\theta = \theta(s)$: an angle function



The Frenet equation

$$\frac{\partial}{\partial s} F = F \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix},$$

$$F = (\mathbf{T}, \mathbf{N}),$$

$\kappa = \frac{\partial \theta}{\partial s}$: a curvature

SOLITON EQUATIONS AND MOTION OF A CURVE

Consider the time evolution

$$\frac{\partial}{\partial t}\gamma(s,t) = g(s,t)\mathbf{T}(s,t) + f(s,t)\mathbf{N}(s,t)$$

A non-stretching condition (isoperimetric condition) $g_s = f\kappa$

↓

$$\frac{\partial}{\partial t}F = F\begin{bmatrix} 0 & -f_s - g\kappa \\ f_s + g\kappa & 0 \end{bmatrix}$$

The compatibility condition gives $\kappa_t = (f_s + g\kappa)_s$

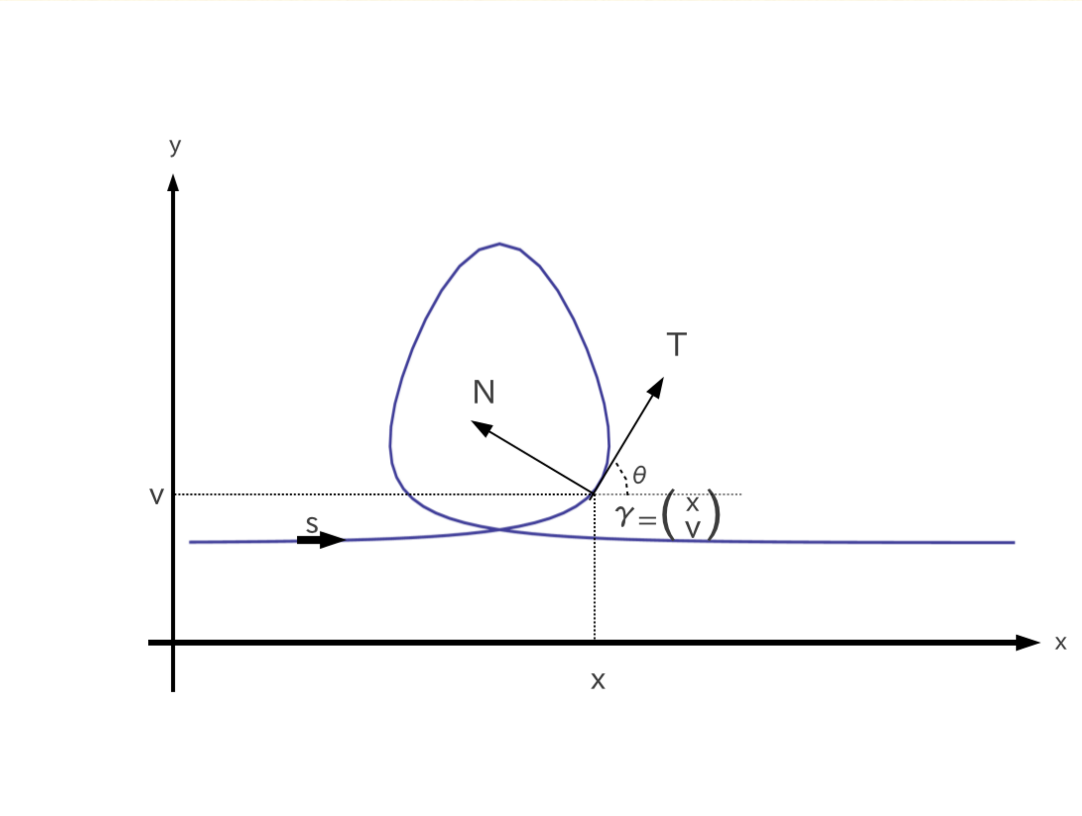
Set $f = -\kappa_s \to g = -\frac{\kappa^2}{2} \to \kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0.$

mKdV equation

SOLITON EQUATIONS AND MOTION OF A CURVE

Describe the motion of a curve in the Cartesian coordinates (x,v)
(an Eulerian description of the motion of a curve):

$$\gamma(s,t) = \left[\begin{array}{c} x(s,t) \\ v(s,t) \end{array} \right] = \int \left[\begin{array}{c} \cos\theta(s,t) \\ \sin\theta(s,t) \end{array} \right] ds$$



In the mKdV curve,
write down geometric quantities
in terms of v, x, t.

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}} \right)_x$$

(potential) WKI elastic beam equation

SOLITON EQUATIONS AND MOTION OF A DISCRETE CURVE

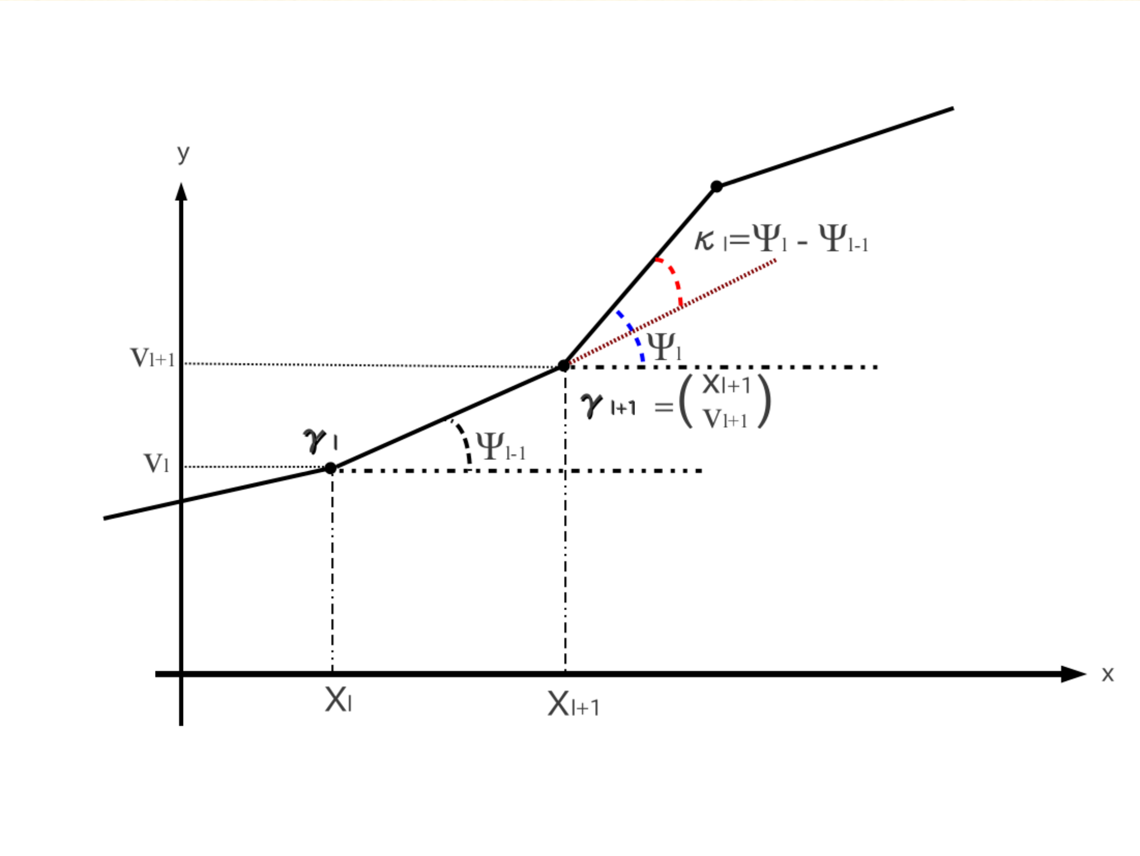
Doliwa & Santini 1995, Inoguchi-Kajiwara-Matsuura-Ohta 2011

A discrete curve $\gamma_l = \gamma(s_l)$

Tangent vector $\mathbf{T}_l = \frac{\gamma_{l+1}-\gamma_l}{a_l}, \left|\frac{\gamma_{l+1}-\gamma_l}{a_l}\right| = 1.$

$$s_l = \sum_{k=0}^{l-1} a_k$$

$$\frac{\gamma_{l+1} - \gamma_l}{a_l} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}$$



The discrete Frenet equation

$$F_{l+1} = F_l \begin{bmatrix} \cos\kappa_l & -\sin\kappa_l \\ \sin\kappa_l & \cos\kappa_l \end{bmatrix}$$

$$F_l = (\mathbf{T}_l, \mathbf{N}_l)$$

$$\mathbf{T}_{l+1} \cdot \mathbf{T}_l = \cos\kappa_l$$

SOLITON EQUATIONS AND MOTION OF A CURVE

Consider the time evolution

$$\frac{\partial}{\partial t}\gamma_l = g_l \mathbf{T}_l + f_l \mathbf{N}_l$$

A non-streching (isoperimetric) condition $g_{l+1}\cos\kappa_l - g_l = f_{l+1}\sin\kappa_l$

$$\frac{\partial}{\partial t}F = F\begin{bmatrix} 0 & \underbrace{\frac{g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l}{a_l}}_{A_l} \\ -\frac{g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l}{a_l} & 0 \end{bmatrix}$$

The compatibility condition gives $\frac{d\kappa_l}{dt} = \frac{A_{l+1}}{a_{l+1}} - \frac{A_l}{a_l}$

Set $f_l = -u_{l-1} = -\tan\frac{\kappa_n}{2}$, $g_l = 1$, $a_l = a \rightarrow \frac{du_l}{dt} = \frac{1}{2a}(1+u_l^2)(u_{l+1} - u_{l-1})$

semi-discrete mKdV equation

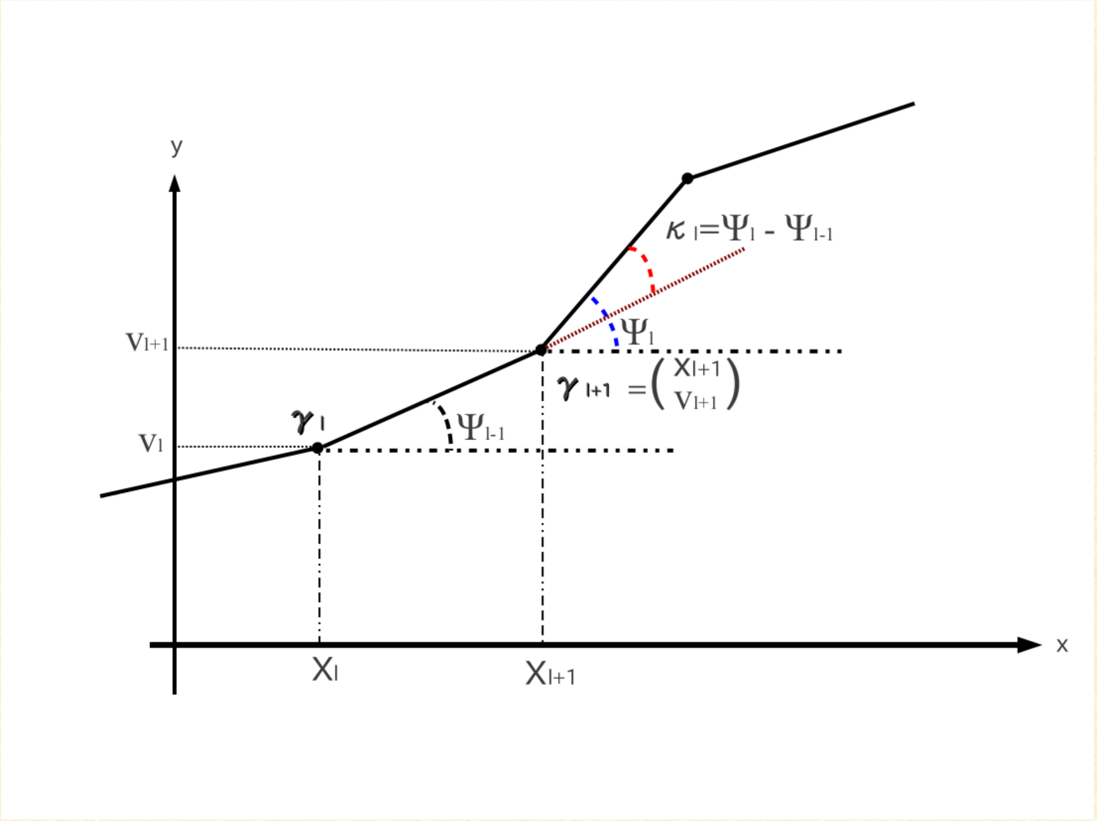
EXPRESSION IN THE CARTESIAN COORDINATES

Describe the motion of a discrete curve in the Cartesian coordinates

$$(X_l, v_l)$$

(an Eulerian description of the motion of a discrete curve):

$$\gamma_l(t) = \begin{bmatrix} X_l(t) \\ v_l(t) \end{bmatrix} = \sum_{j=0}^{l-1} \begin{bmatrix} \epsilon\cos(\psi_j) \\ \epsilon\sin(\psi_j) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}$$



In the semi-discrete mKdV curve, write down geometric quantities in terms of v_l, X_l, t

$$\frac{dX_l}{dt} = \frac{X_{l+1} - X_l}{a} + \frac{v_{l+1} - v_l}{a}\,\frac{v_{l+1} - 2v_l + v_{l-1}}{X_{l+1} - 2X_l + X_{l-1}},$$

$$\frac{dv_l}{dt} = \frac{v_{l+1} - v_l}{a} + \frac{X_{l+1} - X_l}{a}\,\frac{v_{l+1} - 2v_l + v_{l-1}}{X_{l+1} - 2X_l + X_{l-1}}.$$

The semi-discrete (potential) WKI elastic beam equation

SELF-ADAPTIVE MOVING MESH SCHEME OF THE WKI ELASTIC BEAM EQUATION

The system can be rewritten into

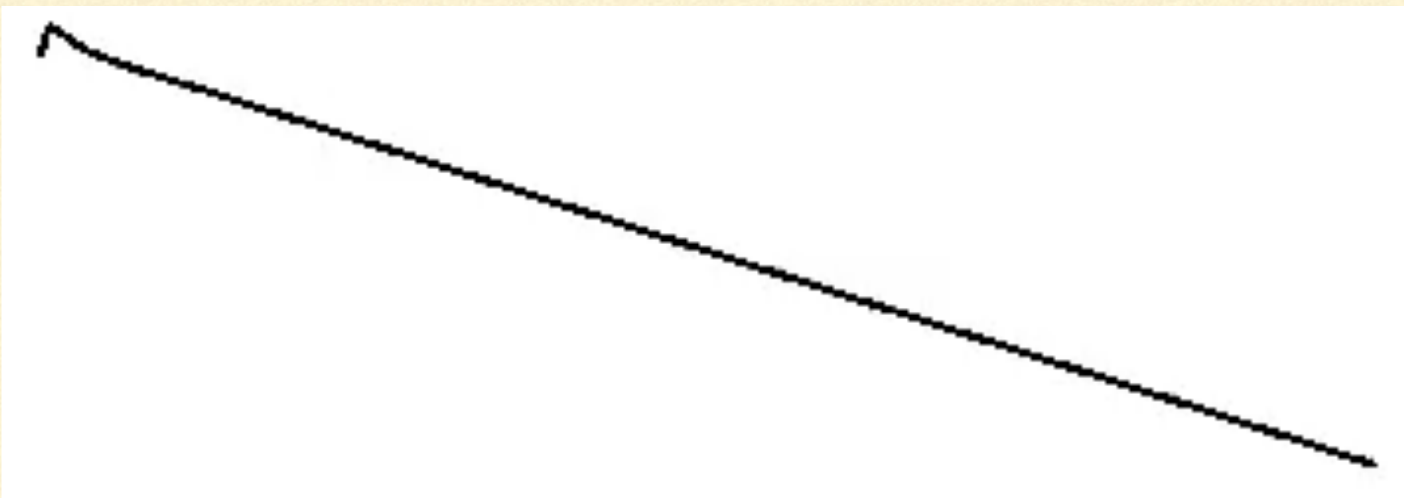
$$\frac{d\delta_l}{dt} = -\frac{v_{l+1} - v_l}{a}\left(G_{l+1} + G_l\right),$$

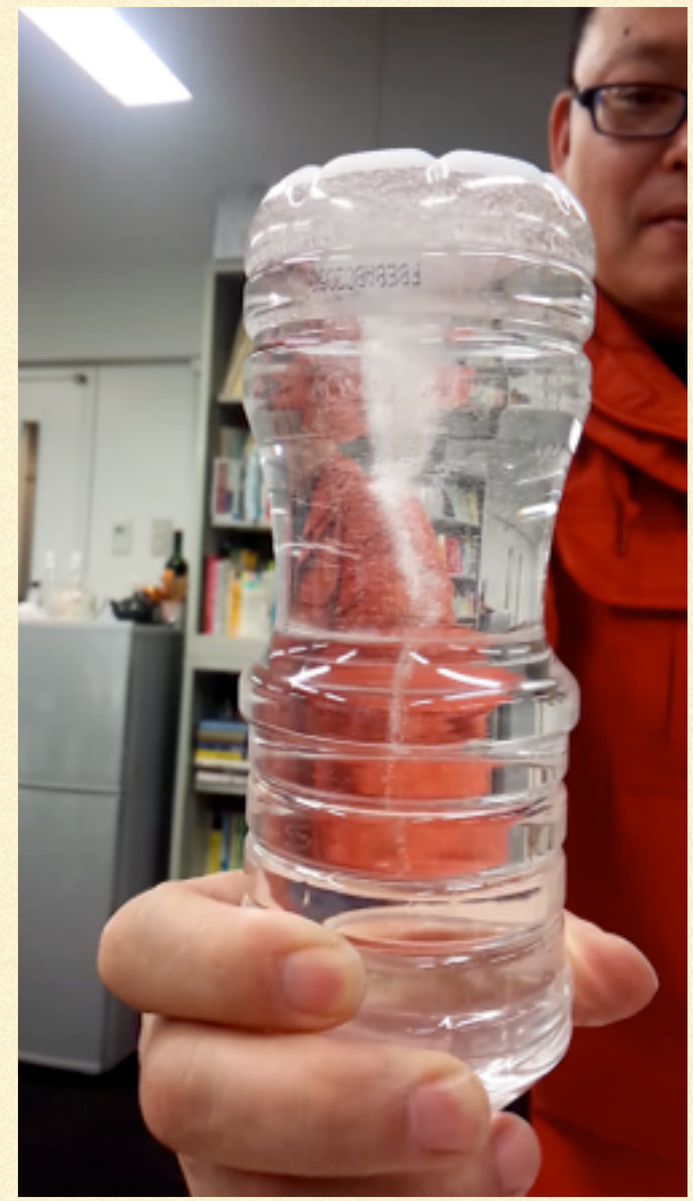
$$\frac{d}{dt}\left(v_{l+1} - v_l\right) = \frac{\delta_l}{a}\left(G_{l+1} + G_l\right),$$

$$\delta_l \equiv X_{l+1} - X_l, \quad G_l \equiv \frac{v_{l+1} - 2v_l + v_{l-1}}{\delta_l - \delta_{l-1}}$$

Feng-Inoguchi-Kajiwara-KM-Ohta 2011

VORTEX FILAMENT





Local induction approximate (LIA) equation
(Da Rios 1906)

unit binormal vector

$$\frac{\partial \mathbf{X}}{\partial t} = \frac{\Gamma}{4\pi}\left[\log\left(\frac{L}{\sigma}\right)\right]\kappa\mathbf{B}$$

curvature

renormalize

$$\mathbf{X_t} = \kappa\mathbf{B} \quad \longrightarrow \quad \mathbf{X_t} = \mathbf{X_s} \times \mathbf{X_{ss}}$$

$\mathbf{X}$: a position vector on a vortex filament

VORTEX FILAMENT IN NATURE

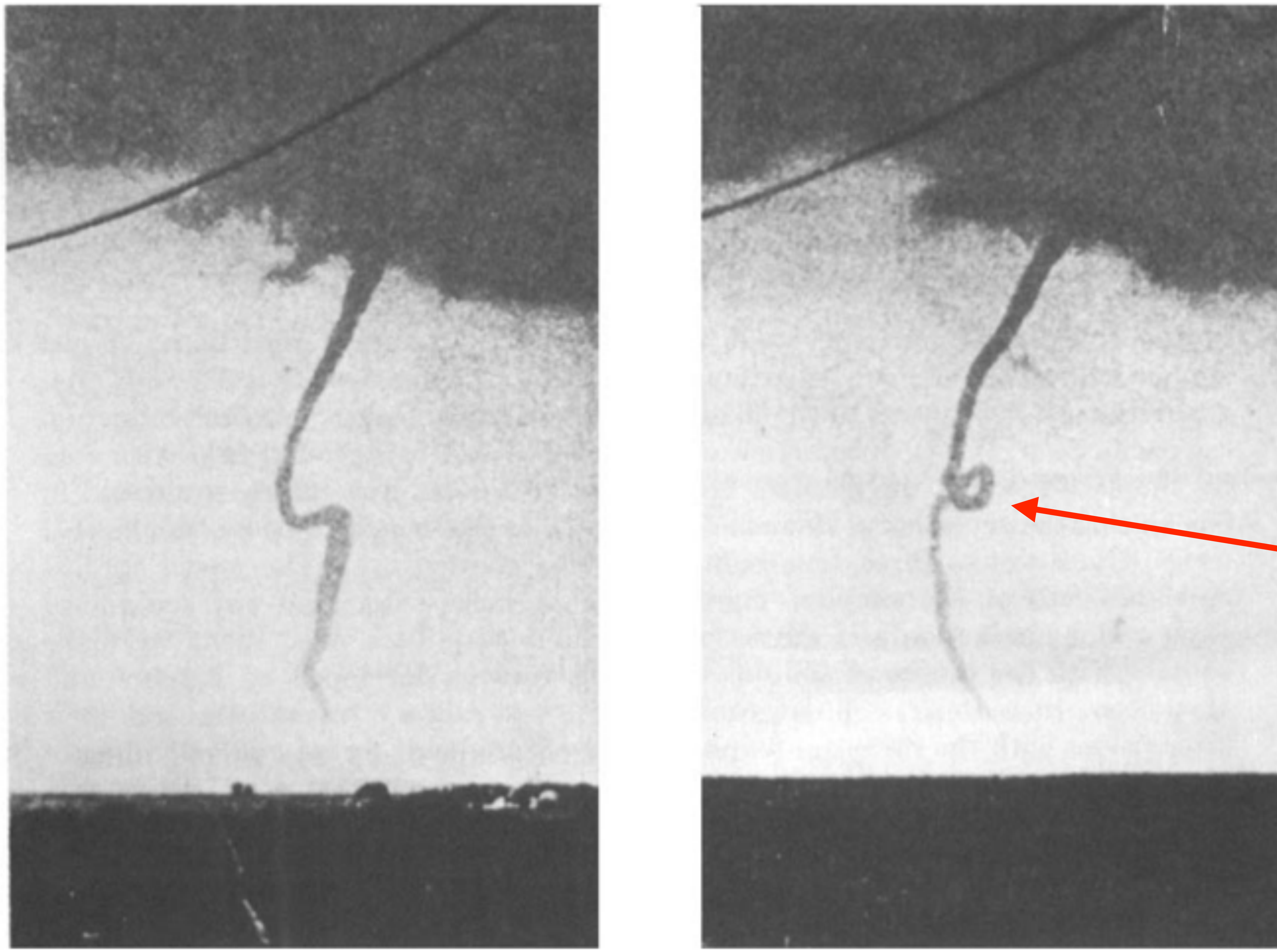


FIGURE 1. Two photographs of a tornado near Braman, Oklahoma (11 May 1978) showing a distinct large-amplitude localized twist of the vortex core. Photographs by T. Goggin, *Newkirk Herald Journal* (reproduced with permission).

Aref & Flinchem
JFM 1984

soliton

NUMERICAL SCHEME

Aref & Flinchem, JFM 1984

$$\frac{d\mathbf{X}_n}{dt} = \frac{\mathbf{X}_{n-1} \times \mathbf{X}_n + \mathbf{X}_n \times \mathbf{X}_{n+1} + \mathbf{X}_{n+1} \times \mathbf{X}_{n-1}}{(\Delta s)^3}$$

This scheme does not possess integrability.

PDES DESCRIBING VORTEX FILAMENTS

LIA equation

$$\mathbf{X_t} = \mathbf{X_s} \times \mathbf{X_{ss}}$$

Hashimoto transformation

Nonlinear Schrödinger equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}\left[\,|\psi|^2 + A(t)\,\right]\psi = 0$$

Hashimoto transformation $\psi(s,t) = \kappa(s,t)\exp\left[i\int_0^s \tau\left(s',t\right)\,ds'\right]$

Integrate w.r.t. s

$$\mathbf{T} = \mathbf{X_s}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Gauge Transformation

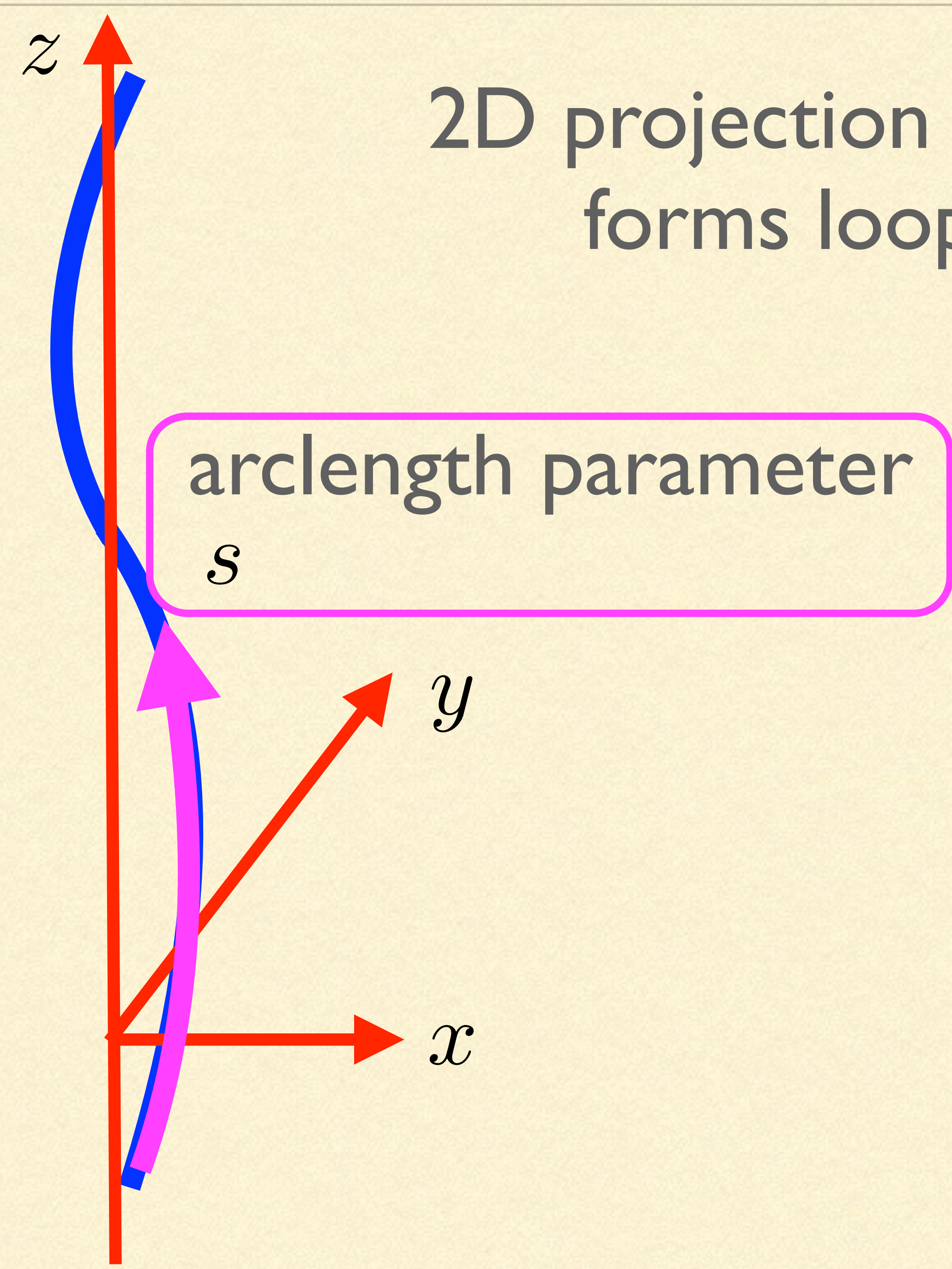
$$\mathbf{T_t} = \mathbf{T} \times \mathbf{T_{ss}}$$

Heisenberg ferromagnet equation

VORTEX FILAMENT

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Cartesian coordinates



2D projection of VF soliton forms loop soliton

PDES DESCRIBING VORTEX FILAMENTS

Hasimoto transformation $\psi(s,t) = \kappa(s,t)\exp\left[i \int_0^s \tau(s',t)\, ds' \right]$

LIA equation

$$\mathbf{X_t} = \mathbf{X_s} \times \mathbf{X_{ss}}$$

Nonlinear Schrödinger equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}\left[\, |\psi|^2 + A(t) \,\right]\psi = 0$$

Integrate w.r.t. s

$$\mathbf{T} = \mathbf{X_s}$$

(Konno-Mitsuhashi-Ichikawa)
Hodograph transformation

$$z = \int \operatorname{sgn}\left(\frac{\partial z}{\partial s}\right)\left(1 - \left|\frac{\partial \Phi}{\partial s}\right|^2\right)^{\frac{1}{2}} ds$$

$$\Phi = x + iy$$

$$\mathbf{T_t} = \mathbf{T} \times \mathbf{T_{ss}}$$

Heisenberg ferromagnet equation

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Complex WKI equation

$$i\Phi_t + \operatorname{sgn}\left(\frac{\partial z}{\partial s}\right)\left(\frac{\Phi_z}{\sqrt{1+|\Phi_z|^2}}\right)_z = 0$$

Semi-discrete NLS and Heisenberg ferromagnet equation

Semi-discrete NLS equation (Ablowitz-Ladik)

$$i\frac{d}{dt}\Psi_n = \frac{\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}}{a^2} + |\Psi_n|^2\left(\Psi_{n+1} + \Psi_{n-1}\right)$$

Semi-discrete Heisenberg ferromagnet equation (Ishimori 1982)

$$\frac{d}{dt}\mathbf{T}_n = \frac{2}{a^2\left(1 + \mathbf{T}_n \cdot \mathbf{T}_{n+1}\right)}\mathbf{T}_n \times \mathbf{T}_{n+1} - \frac{2}{a^2\left(1 + \mathbf{T}_{n-1} \cdot \mathbf{T}_n\right)}\mathbf{T}_{n-1} \times \mathbf{T}_n$$

$$T_n^x T_{n+1}^y - T_n^y T_{n+1}^x = 0$$

$$\mathbf{T}_n = \frac{\mathbf{X}_{n+1} - \mathbf{X}_n}{|\mathbf{X}_{n+1} - \mathbf{X}_n|} = \frac{\mathbf{X}_{n+1} - \mathbf{X}_n}{a},$$

$$\mathbf{X}_n = x_n\mathbf{e}_x + y_n\mathbf{e}_y + z_n\mathbf{e}_z,$$

SPACE DISCRETIZATION OF THE COMPLEX WKI EQUATION

Set $\Phi_n = x_n + iy_n$

$$\frac{d}{dt}\Phi_n = \frac{2i\left[(z_n - z_{n-1})\left(\Phi_{n+1} - \Phi_n\right) - \left(\Phi_n - \Phi_{n-1}\right)(z_{n+1} - z_n)\right]}{\left[a^2 + (\mathbf{X}_n - \mathbf{X}_{n-1}) \cdot (\mathbf{X}_{n+1} - \mathbf{X}_n)\right]}$$

$$\frac{d}{dt}z_n = \frac{2\left[(x_n - x_{n-1})(y_{n+1} - y_n) - (y_n - y_{n-1})(x_{n+1} - x_n)\right]}{\left[a^2 + (\mathbf{X}_n - \mathbf{X}_{n-1}) \cdot (\mathbf{X}_{n+1} - \mathbf{X}_n)\right]}$$

Self-adaptive moving mesh scheme for a vortex filament

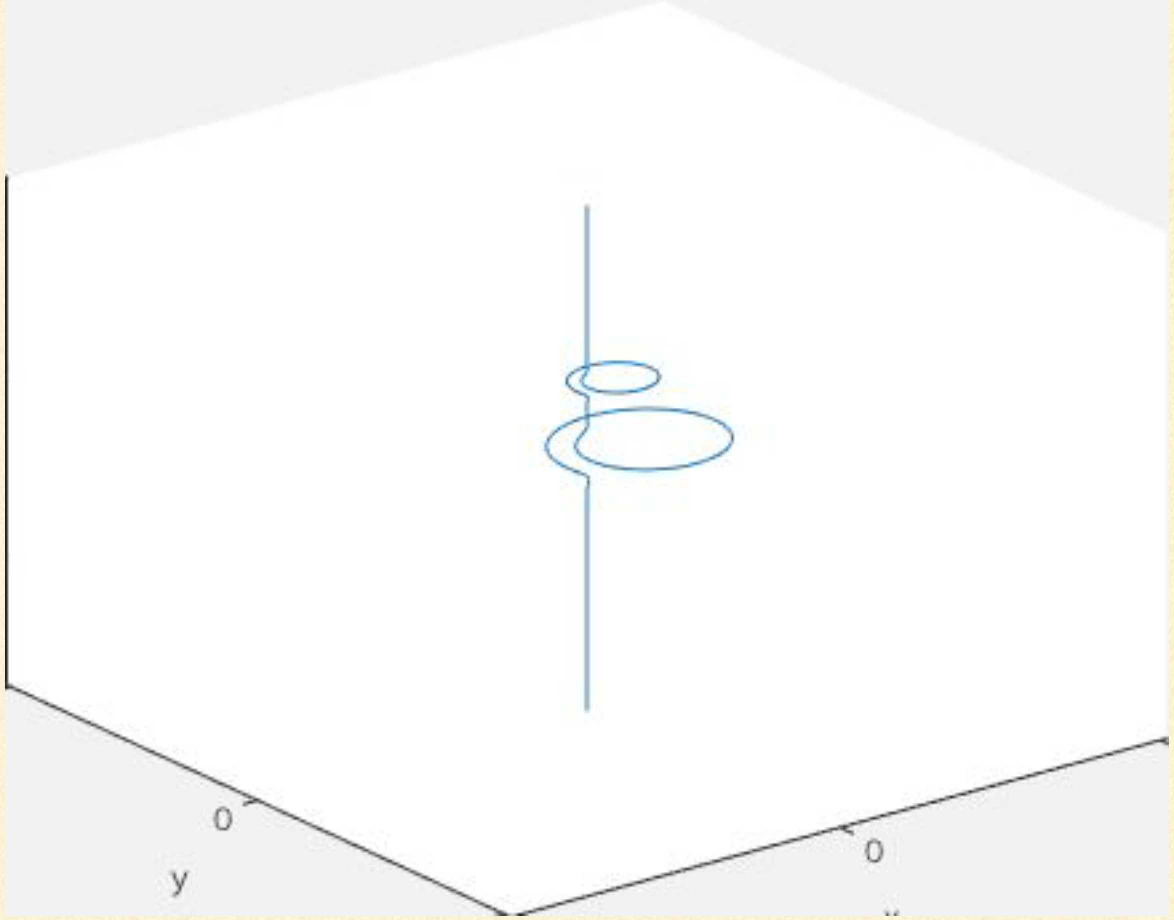
dependent variable(complex)

$\Phi_n = x_n + iy_n$

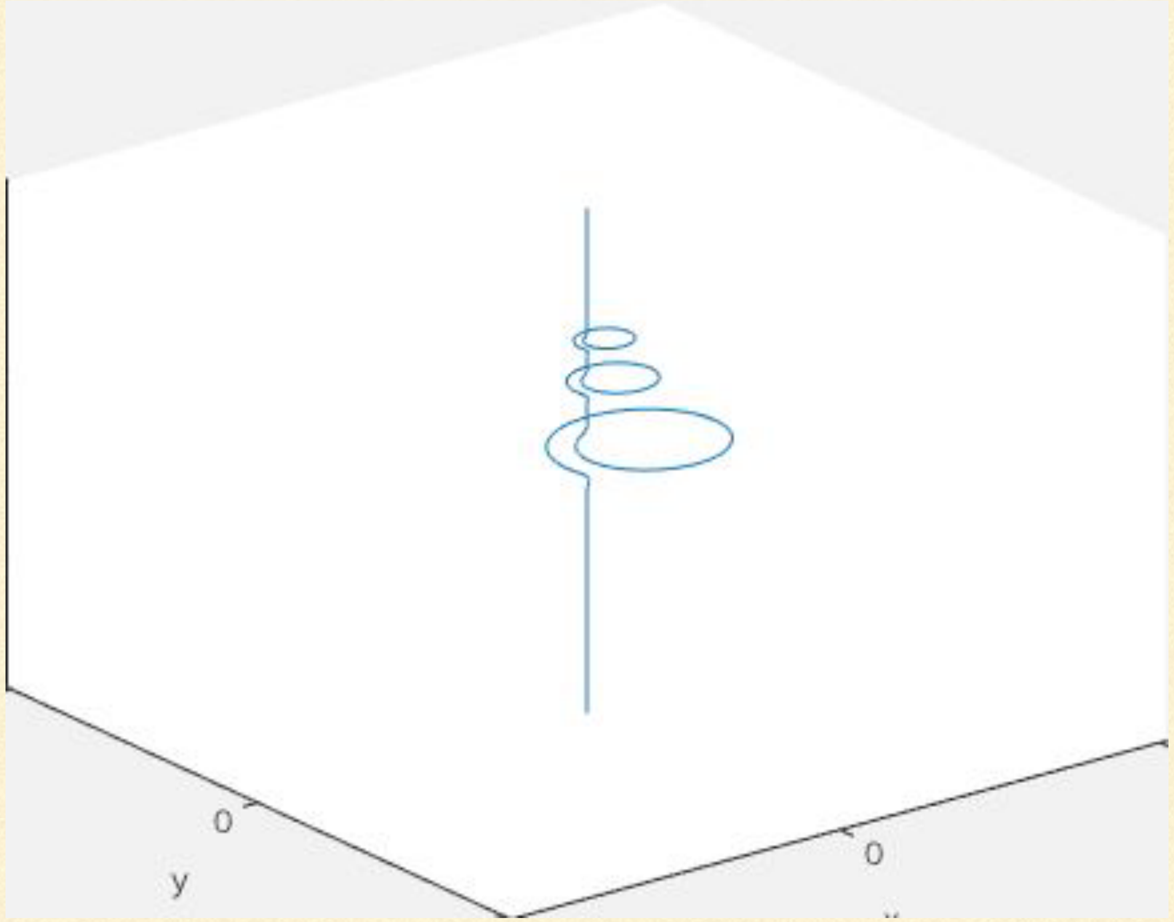
independent variable(mesh point)

z_n

SOLITON ON A VORTEX FILAMENT (NUMERICAL RESULT)



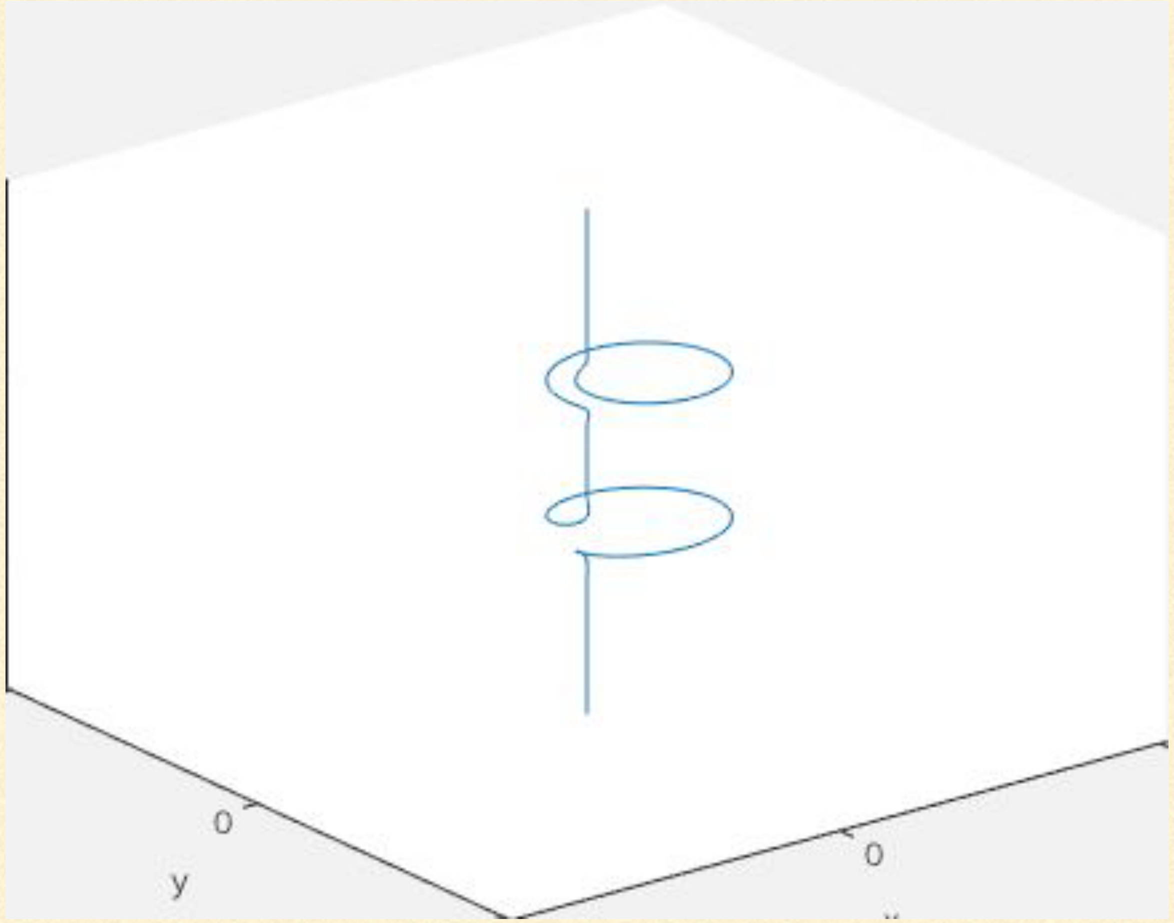
2 vortex solitons



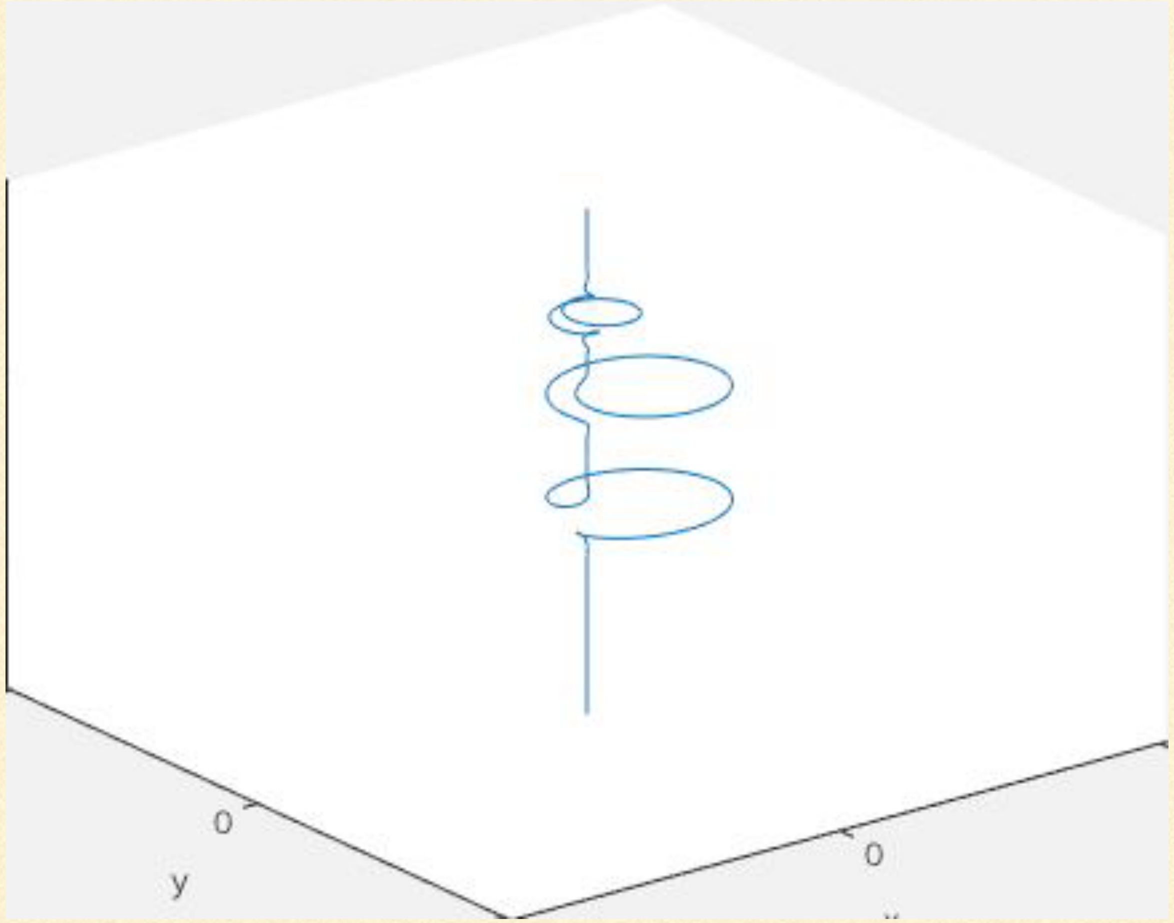
3 vortex solitons

Time discretization: Leapfrog method

SOLITON ON A VORTEX FILAMENT (NUMERICAL RESULT)

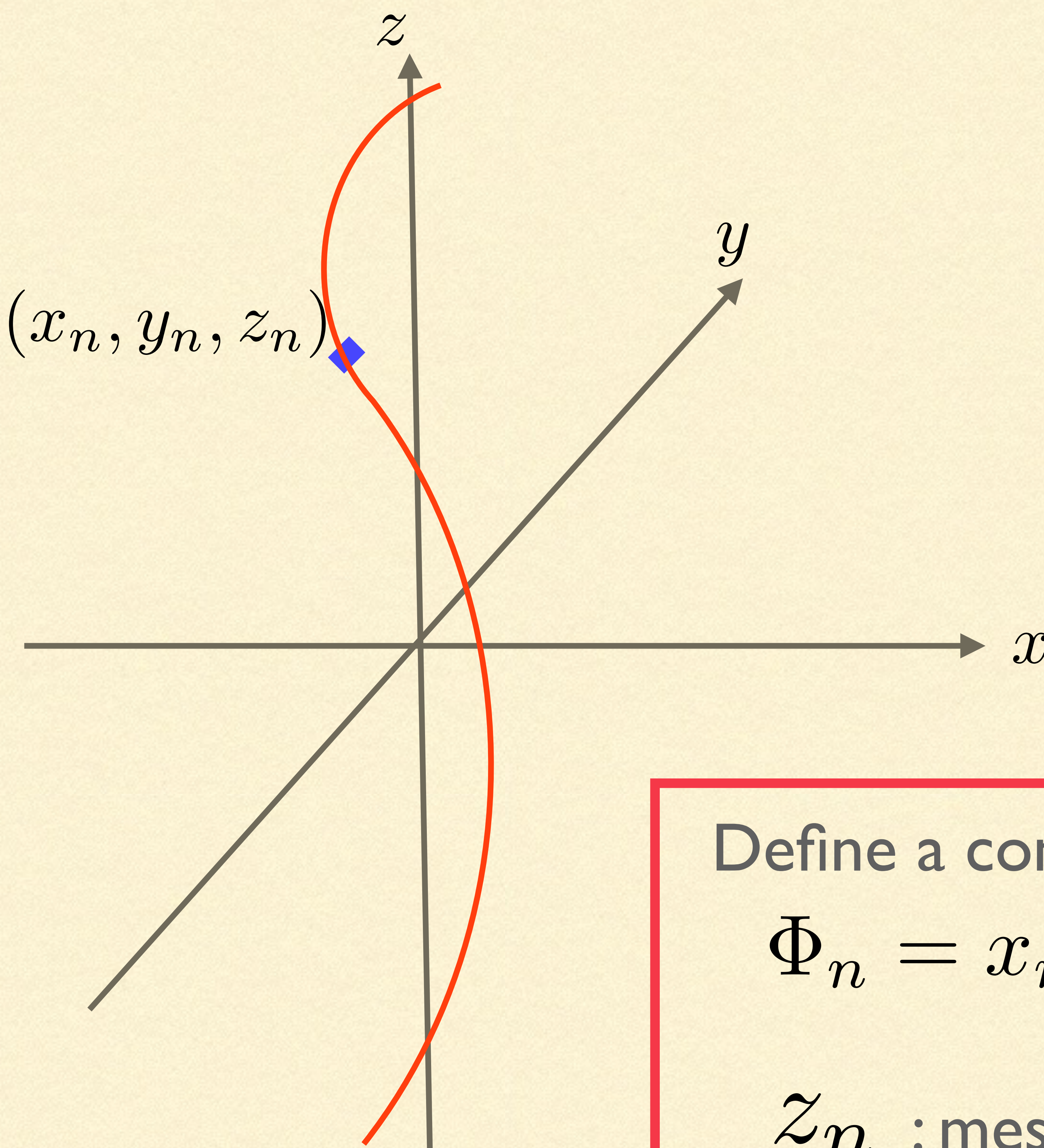


2 vortex solitons: head-on collision



3 vortex solitons

How to make self-adaptive moving mesh schemes in 3D curve



Define a complex function

$$\Phi_n = x_n + iy_n$$

z_n : mesh point

COMPLEX SHORT PULSE EQUATION

$$u_{zt} = u + \frac{1}{2}(|u|^2 u_z)_z$$

↓ Space discretization

$$\partial_T \delta_k = \frac{-|u_{k+1}|^2 + |u_k|^2}{2}$$

$$\partial_T (u_{k+1} - u_k) = \delta_k \frac{u_{k+1} + u_k}{2}$$

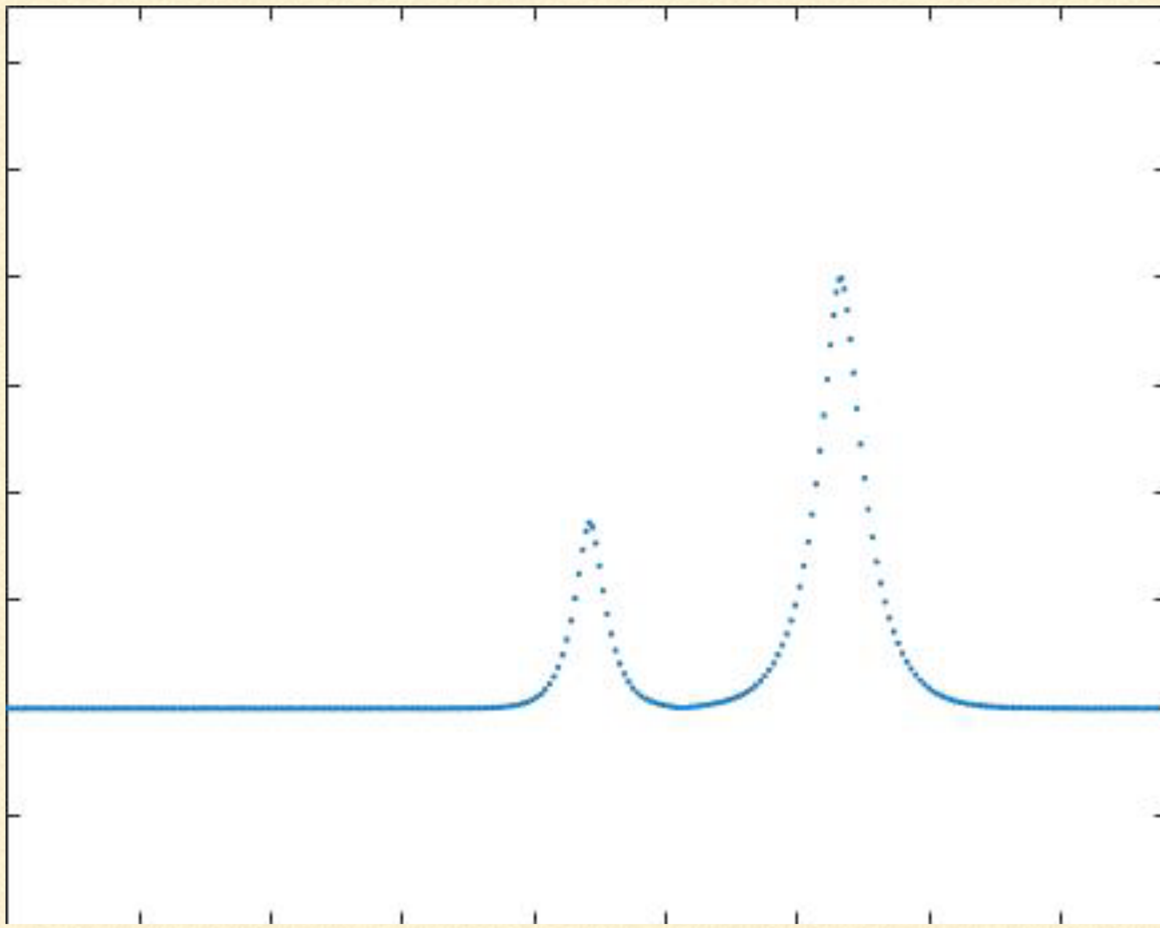
F-M-O 2014

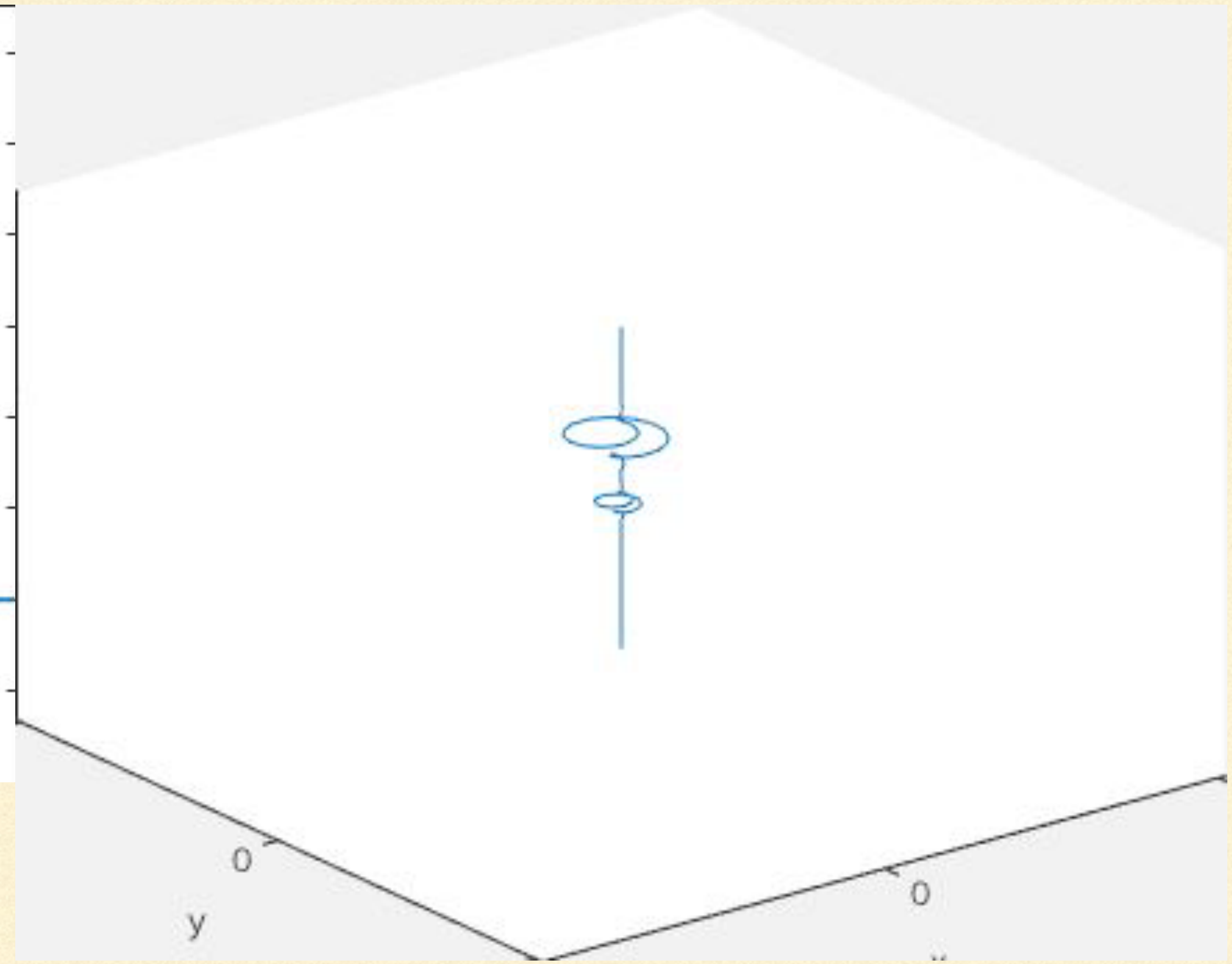
Set

$$u_k = x_k + i y_k, \quad \delta_k = z_{k+1} - z_k$$

↓

NUMERICAL SIMULATION FOR COMPLEX SP EQUATION





Complex SP can be interpreted as an equation of motion of curve in 3D space

SUMMARY

- We found a systematic method to create self-adaptive moving mesh schemes in 3-dimension.
- Keys to construct self-adaptive moving mesh schemes: Conservation laws and hodograph transformations.
- Future problems: Construction of self-adaptive moving mesh schemes for (2+1)-dimensional PDEs.