Special solutions of integrable equations and reduction technique

What are integrable equations?

Korteweg-de Vries eq (KdV eq)

$$u_t + u u_x + u_{xxx} = 0$$

modified KdV eq.

$$u_t + u^2 u_x + u_{xxx} = 0$$

nonlinear Schrödinger eq

$$i u_t = u_{xx} + |u|^2 u$$

sine-Gordon eq.

$$u_{xx} + u_{yy} = \sin u$$

Toda lattice eq.

$$\frac{d^2}{dt^2}\log u_n = u_{n+1} - 2u_n + u_{n-1}$$

discrete KdV eq.

$$u_n^{t+1} - u_n^{t-1} = \frac{1}{u_{n+1}^t} - \frac{1}{u_{n-1}^t}$$

Kadomtsev-Petviashvili eq (KP eq)

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0$$

Davey-Stewartson eq.

$$iu_t = u_{xx} - u_{yy} + (|u|^2 - 2\phi_x)u$$

$$\phi_{xx} + \phi_{yy} = (|u|^2)_x$$

Painlevé eq (6 types)

PI : $u_{xx} = 6u^2 + x$

PII : $u_{xx} = 2u^3 + xu + a$

PIII : $u_{xx} = \dfrac{u_x^2}{u} - \dfrac{u_x}{u} + \dfrac{u^2}{x^2}(au+b) + \dfrac{c}{x} + \dfrac{d}{u}$

$\vdots$

PVI

$(a, b, c, d : \text{const})$

Nonlinear differencial / difference equations with rich structures

- conservation laws, symmetries
- Hamiltonian structure
- Lax pair
- Bäcklund transformation
- inverse scattering transformation
  initial/boundary value problem
- special solutions

Why integrable equations?

in physics

explicit description of physical phenomena, quantities

in numerical analysis

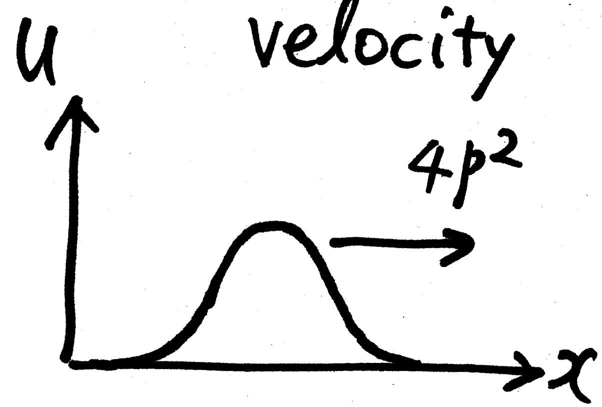
numerical scheme $\approx$ discrete equation

- acceleration of convergence of series
- diagonalization of matrix
- self-adaptive moving mesh

Special solutions

KdV eq. $\quad u_t + u u_x + u_{xxx} = 0$

typical solutions:



- 1-soliton

$$u = 12p^2 \operatorname{sech}^2(px - 4p^3 t)$$

(p: const)

- N-soliton

nonlinear superposition

Painlevé eq ($P_{\mathrm{II}} \sim P_{\mathrm{VI}}$)

typical solutions:

- Riccati solutions (special functions)
- rational solutions

Reduction technique (dimensional reduction)

KP $\quad (u_t + u u_x + u_{xxx})_x + u_{yy} = 0$

$\Downarrow \quad u(x,y,t) = u(x,t) \quad$ y-independent

KdV $\quad u_t + u u_x + u_{xxx} = 0$

sine-Gordon $\quad (\partial_x^2 + \partial_y^2) u = \sin u$

$\Downarrow \quad u(r\cos\theta, r\sin\theta) = u(r) \quad$ θ-independent

Painlevé III $\quad (\partial_r^2 + \frac{1}{r}\partial_r) u = \sin u$

Reduction of solutions

Special solution of soliton eqs.

$\Downarrow$ reduction

Special solution of Painlevé eqs.

Riccati, rational

mKP $\longrightarrow$ cylindrical mKdV $\longrightarrow$ PII

modified KP eq

$$4u_t = (u_{xx} + 4u^3 + 6u\phi_x + 3\phi_y)_x$$

$$u_y = (u^2 + \phi_x)_x$$

cylindrical mKdV eq

$$4u_t = \left(u_{xx} - 2u^3 - 2\frac{x}{t}u\right)_x$$

Painlevé II eq

$$u_{xx} = 2u^3 - 2xu + a \qquad (a : \text{const})$$

mKP

$$4u_t = (u_{xx} + 4u^3 + 6u\phi_x + 3\phi_y)_x$$

$$u_y = (u^2 + \phi_x)_x$$

$$\Downarrow \quad u(x,y,t) = v\left(x - \frac{y^2}{3t}, t\right) - \frac{y}{3t}$$

$$\phi(x,y,t) = \psi\left(x - \frac{y^2}{3t}, t\right) - \frac{y}{3t}$$

$$\xi = x - \frac{y^2}{3t} \qquad \tau = t$$

cylin. mKdV

$$4v_\tau = \left(v_{\xi\xi} - 2v^3 - 2\frac{\xi}{\tau}v\right)_\xi$$

cylin. mKdV

$$4v_\tau = \left(v_{\xi\xi} - 2v^3 - 2\frac{\xi}{\tau}v\right)_\xi$$

$$\Downarrow \quad v(\xi,\tau) = -\frac{1}{(3\tau)^{1/3}}\, q\left(-\frac{\xi}{(3\tau)^{1/3}}\right)$$

$$z = -\frac{\xi}{(3\tau)^{1/3}}$$

P II

$$q_{zz} = 2q^3 - 2zq + a \qquad (a:\text{const})$$

Special solutions of mKP:

soliton solutions (exponential function)

$$4u_t = (u_{xx} + 4u^3 + 6u\phi_x + 3\phi_y)_x$$

$$u_y = (u^2 + \phi_x)_x$$

$$u = \left(\log \frac{g}{f}\right)_x \qquad \phi = (\log gf)_x$$

$$(D_x^2 - D_y)\, g \cdot f = 0$$

$$(D_x^3 + 3D_x D_y - 4D_t)\, g \cdot f = 0$$

D_x, D_y, D_t : Hirota bilinear operator

Hirota bilinear operator

$$D_x g\cdot f = g_x f - g f_x$$

$$D_x^2 g\cdot f = g_{xx} f - 2 g_x f_x + g f_{xx}$$

$$D_x^3 g\cdot f = g_{xxx} f - 3 g_{xx} f_x + 3 g_x f_{xx} - g f_{xxx}$$

$$D_x D_y g\cdot f = g_{xy} f - g_x f_y - g_y f_x + g f_{xy}$$

$$P(D_x, D_y, D_t)\, g\cdot f$$

$$= P(\partial_x - \partial_X, \partial_y - \partial_Y, \partial_t - \partial_T)\big(g(x,y,t) f(X,Y,T)\big)\Big|_{\substack{X=x\\ Y=y\\ T=t}}$$

$$f = \det\left(\varphi_i^{(j)}\right)_{1\le i,\, j\le N}$$

$$g = \det\left(\varphi_i^{(j)}\right)_{1\le i,\, j\le N+1}$$

(Wronskian in x)

$$\boxed{\partial_x \varphi_i^{(j)} = \varphi_i^{(j+1)}, \quad \partial_y \varphi_i^{(j)} = \varphi_i^{(j+2)}, \quad \partial_t \varphi_i^{(j)} = \varphi_i^{(j+3)}}$$

(linear dispersion relation)

$$\varphi_i^{(j)} = \alpha_i p_i^j e^{p_i x + p_i^2 y + p_i^3 t} + \beta_i q_i^j e^{q_i x + q_i^2 y + q_i^3 t}$$

($\alpha_i, \beta_i, p_i, q_i$: const)

Special solutions of PII :

Riccati solutions (Airy function)

$$q_{zz} = 2q^3 - 2zq + a$$

$$q = \left(\log \frac{G}{F}\right)_z$$

$$(D_z^2 - z)\, G \cdot F = 0$$

$$(D_z^3 - z D_z - a)\, G \cdot F = 0$$

For $a = 2N+1$ $(N = 0, 1, 2, \cdots)$

$$F = \det\left(\partial_z^{i+j-2} A\right)_{1 \le i, j \le N}$$

$$G = \det\left(\partial_z^{i+j-2} A\right)_{1 \le i, j \le N+1}$$

$A = A(z)$: Airy function

$$\partial_z^2 A(z) = z A(z)$$

Hankel determinants (Wronskian in horizontal and vertical)

For $N=0$

$$a=1 \qquad q_{zz} = 2q^3 - 2zq + 1$$

$$q = \left(\log\frac{G}{F}\right)_z \qquad F=1, \quad G=A$$

$$(D_z^2 - z)\,G\cdot F = 0 \iff (\partial_z^2 - z)\,G = 0$$

$$\Big\downarrow \partial_z$$

$$(D_z^3 - zD_z - 1)\,G\cdot F = 0$$

$$\iff (\partial_z^3 - z\partial_z - 1)\,G = 0$$

linear differential eq

$$\text{mKP} \qquad \text{exponential func.}$$

$$e^{px+p^2y+p^3t}$$

$$\downarrow$$

cylin. mKdV

$$\downarrow$$

$$\text{P II} \qquad \text{Airy func.}$$

$$A(z)$$

mKP $\longrightarrow$ cylin. mKdV

$$u(x,y,t) = v(\xi,\tau) - \frac{y}{3t} \qquad \xi = x - \frac{y^2}{3t}, \quad \tau = t$$

cylin. mKdV $\longrightarrow$ PII

$$v(\xi,\tau) = -\frac{1}{(3\tau)^{1/3}} q(z) \qquad z = -\frac{\xi}{(3\tau)^{1/3}}$$

$$u(x,y,t) = \left(\log\frac{g}{f}\right)_x \qquad f, g = \det\left(\varphi_i^{(j)}\right)$$

$$\boxed{\begin{aligned} \partial_y \varphi_i^{(j)} &= \partial_x^2 \varphi_i^{(j)} \\ \partial_t \varphi_i^{(j)} &= \partial_x^3 \varphi_i^{(j)} \end{aligned}}$$

$$\varphi_i^{(j)} = \int p^j e^{px+p^2y+p^3t} w(p)\,dp$$

exponent of mKP solution

$$px+p^2y+p^3t$$

$$=\left(p+\frac{y}{3t}\right)\left(x-\frac{y^2}{3t}\right)+\left(p+\frac{y}{3t}\right)^3t-\frac{xy}{3t}+\frac{2y^3}{27t^2}$$

$$\int e^{px+p^2y+p^3t}\,dp$$

$$=e^{-\frac{xy}{3t}+\frac{2y^3}{27t^2}}\int e^{\left(p+\frac{y}{3t}\right)\left(x-\frac{y^2}{3t}\right)+\left(p+\frac{y}{3t}\right)^3t}\,dp$$

integral representation of Airy function

$$A(z)=\int_C e^{zp-\frac{1}{3}p^3}\,dp$$

cylin. mKdV $\quad 4v_\tau = \left(v_{\xi\xi} - 2v^3 - 2\frac{\xi}{\tau}v\right)_\xi$

$$v = \left(\log \frac{\mathcal{G}}{\mathcal{F}}\right)_\xi \qquad \mathcal{F} = \det\left(\partial_z^{j-1} A\left(-\frac{\xi - \xi_i}{(3\tau)^{1/3}}\right)\right)_{1\le i,j\le N}$$

$$\mathcal{G} = \det\left(\quad " \quad \right)_{1\le i,j\le N+1}$$

$(\xi_i : \text{const})$

$\Downarrow \ \xi_i \to 0$

cylin. mKdV $\longrightarrow$ PII

$$v(\xi,\tau) = -\frac{1}{(3\tau)^{1/3}}\, q\left(-\frac{\xi}{(3\tau)^{1/3}}\right)$$

$$\mathrm{mKP} \longrightarrow \mathrm{mKdV} \longrightarrow \mathrm{P_{II}}$$

mKP
$$\begin{cases} 4u_t = (u_{xx} + 4u^3 + 6u\phi_x + 3\phi_y)_x \\ u_y = (u^2 + \phi_x)_x \end{cases}$$

$$\Downarrow \quad u(x,y,t) = v(x,t), \quad \phi(x,y,t) = \psi(x,t)$$

mKdV
$$4v_t = v_{xxx} - 6u^2 u_x$$

$$\Downarrow \quad v(x,t) = \frac{1}{t^{1/3}} q\left(\frac{x}{t^{1/3}}\right) \qquad z = \frac{x}{t^{1/3}}$$

P II
$$q_{zz} = 2q^3 - \frac{4}{3} z q + a \qquad (a : \mathrm{const})$$

Special solutions of mKP :

rational solutions (Schur polynomial)

$$u = \left(\log \frac{g}{f}\right)_x \qquad \phi = \left(\log gf\right)_x$$

$$f = \det\left(S_{\lambda_i + i - j}\right)_{1 \le i, j \le N} \qquad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N \le \lambda_{N+1}$$

$$g = \det\left(\quad " \quad \right)_{1 \le i, j \le N+1}$$

$$e^{p x_1 + p^2 x_2 + p^3 x_3 + \cdots} = \sum_{n=0}^{\infty} S_n(x_1, x_2, x_3, \cdots) p^n$$

$x_1 = x$, $x_2 = y$, $x_3 = t$, $x_4, x_5, \cdots$: const

$$\text{mKP} \xrightarrow[\text{y-independent}]{} \text{mKdV}$$

rational sols.

$$\Uparrow$$

$\lambda_i = i$ $\quad f = \det\left(S_{2i-j}\right)_{1\le i,j\le N}$

$\quad g = \det\left(\quad " \quad \right)_{1\le i,j\le N+1}$

Wronskian in x (horizontal)

" in y (vertical)

$\partial_y f = 0 \quad \partial_y g = 0$

($x_2, x_4, x_6, \cdots$ -independent)

$$\text{mKdV} \longrightarrow \text{PII}$$

$$v(x,t) = \frac{1}{t^{1/3}} q\left(\frac{x}{t^{1/3}}\right) \qquad \text{rational sols.}$$

$$\Uparrow$$

$$x_5 = x_7 = x_9 = \cdots = 0$$

Assuming weight$(x) = 1$ and weight$(t) = 3$,

f and g are homogeneous weight polynomials

$$\text{weight}(f) = \frac{N(N+1)}{2} \qquad \text{weight}(g) = \frac{(N+1)(N+2)}{2}$$

$$t^{-\frac{N(N+1)}{6}} f, \; t^{-\frac{(N+1)(N+2)}{6}} g \; : \text{polynomials in } \frac{x}{t^{1/3}}$$

- Classical solutions of Painlevé eqs. are obtained from solitonic solutions of KP through reduction.
- In the intermediate step of reduction, 1+1 dimensional equations and their special solutions are constructed.