The Cauchy-Riemann Equations in Complex Projective Spaces

Mei-Chi Shaw

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556

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National Taiwan University
Outline

1. The Cauchy-Riemann equations
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   - The Cauchy-Riemann equations in Stein manifolds
   - Tangential Cauchy-Riemann equations

2. The $\overline{\partial}$ operator in $\mathbb{C}P^n$
   - Levi-flat hypersurfaces
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   - $\overline{\partial}$ on pseudoconcave domains in $\mathbb{C}P^n$
   - Nonexistence of Levi-flat hypersurfaces in $\mathbb{C}P^n$

3. Open problems
Existence and boundary regularity on pseudoconvex domains

The $\bar{\partial}$ operator on a domain $\Omega$ in a complex manifold:

- $\bar{\partial}u = f$ in $\Omega$ where $f$ is a $(p, q)$-form satisfying
- $\bar{\partial}f = 0$ in $\Omega$.

**Theorem (Kohn 1963-64)** Suppose that $\Omega$ is a bounded strongly pseudoconvex domain in a complex hermitian manifold with smooth boundary $b\Omega$. For any $f \in C^{\infty}_{(p,q)}(\Omega)$ (or $f \in W^{s}_{(p,q)}(\Omega)$), where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\bar{\partial}f = 0$ in $\Omega$, there exists $u \in C^{\infty}_{(p,q-1)}(\Omega)$ (or $u \in W^{s+\frac{1}{2}}_{(p,q)}(\Omega)$), satisfying $\bar{\partial}u = f$.

Kohn established the subelliptic estimates for the $\bar{\partial}$-Neumann problem:

Let $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, we have

$$\|f\|_{W^{\frac{1}{2}}}^2 \leq C(\|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|f\|^2).$$
Existence and boundary regularity on pseudoconvex domains

**Theorem. (Hörmander 1965)** Let \( \Omega \subset \subset \mathbb{C}^n \) (or a Stein manifold) be a bounded pseudoconvex domain. For any \( f \in L^2_{(p,q)}(\Omega) \), where \( 0 \leq p \leq n \) and \( 1 \leq q < n \), such that \( \overline{\partial} f = 0 \) in \( \Omega \), there exists \( u \in L^2_{(p,q-1)}(\Omega) \) satisfying \( \overline{\partial} u = f \) and

\[
\int_{\Omega} |u|^2 \leq \frac{e^{\delta^2}}{q} \int_{\Omega} |f|^2
\]

where \( \delta \) is the diameter of \( \Omega \).

**Theorem (Kohn 1973)** Suppose that \( \Omega \) is a bounded pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary \( b\Omega \). For any \( f \in C^\infty_{(p,q)}(\overline{\Omega}) \), where \( 0 \leq p \leq n \) and \( 1 \leq q < n \), such that \( \overline{\partial} f = 0 \) in \( \Omega \), there exists \( u \in C^\infty_{(p,q-1)}(\overline{\Omega}) \) satisfying \( \overline{\partial} u = f \).

In \( \mathbb{C}^n \) (or a Stein manifold), there exists a strictly plurisubharmonic function.
Let $M$ be the boundary of a pseudoconvex domain $\Omega$ in $\mathbb{C}^n$. The tangential Cauchy-Riemann equation $\bar{\partial}_b$: 

$$\bar{\partial}_b u = f,$$ 

where $f$ is a $(0,q)$-form on $M$.

Two compatibility conditions:

1. $1 \leq q < n - 1$, 
   $$\bar{\partial}_b f = 0.$$ 

2. $q = n - 1$ 
   $$\int_M f \wedge h = 0, \quad h \in L^2_{(n,0)}(M) \cap \text{Ker}(\bar{\partial}_b).$$

When $q = 1$, solvability for $\bar{\partial}_b$ corresponds to different compatibility conditions for $n \geq 3$ and $n = 2$. 

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Let $\bar{\partial}_b^*$ be the adjoint of $\bar{\partial}_b$. We define the $\bar{\partial}_b$-Laplacian:

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b.$$ 

The operator $\Box_b$ is not elliptic. But $\Box_b$ is subelliptic when the boundary is strongly pseudoconvex (or satisfying condition $Y(q)$).

**Theorem (Kohn 1965)** Suppose that $\Omega$ is a strongly pseudoconvex domain in a complex hermitian manifold with smooth compact boundary $b\Omega$. For any $f \in C^\infty_{(p,q)}(b\Omega)$, where $0 \leq p \leq n$ and $1 \leq q < n - 1$, we have

$$\|f\|_{W^1_2}^2 \leq C(\|\bar{\partial}_b f\|_2^2 + \|\bar{\partial}_b^* f\|_2^2 + \|f\|_2^2).$$

In particular, if $f \in W^s_{(p,q)}(b\Omega)$ with $\bar{\partial}_b f = 0$ in $b\Omega$, there exists $u \in W^{s+\frac{1}{2}}_{(p,q-1)}(b\Omega)$ satisfying $\bar{\partial}_b u = f$. 
Theorem  Suppose that $\Omega$ is a pseudoconvex domain in a Stein manifold with smooth compact boundary $b\Omega$. For any $f \in W^s_{(p,q)}(b\Omega)$ with $\bar{\partial}_b f = 0$ in $b\Omega$, where $0 \leq p \leq n$ and $1 \leq q < n - 1$, there exists $u \in W^s_{(p,q-1)}(b\Omega)$ satisfying $\bar{\partial}_b u = f$.

When $q = n - 1$ and we assume the compatibility condition:

$$\int_M f \wedge h = 0, \quad h \in L^2_{(n,0)}(M) \cap \text{Ker}(\bar{\partial}_b).$$

Then the same results hold.

Corollary  The $\bar{\partial}_b$ operator has closed range for all degrees.

The theorem was proved in Shaw (1985) for $q < n - 1$ and Boas-Shaw (1986) for $q = n - 1$. The closed range property was also obtained in Kohn (1986).
Theorem (Shaw 1984) Suppose that $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ is an annulus between two pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ in $\mathbb{C}^n$ with smooth boundary $b\Omega$, where $n \geq 3$. For any $f \in C_{(p,q)}^\infty(\overline{\Omega})$ (or $L^2_{(p,q)}(\Omega)$), where $0 \leq p \leq n$ and $1 \leq q < n - 1$, such that $\overline{\partial}f = 0$ in $\Omega$, there exists $u \in C_{(p,q-1)}^\infty(\overline{\Omega})$ (or $L^2_{(p,q-1)}(\Omega)$) satisfying $\overline{\partial}u = f$.

When $q = n - 1$, the $L^2$ cohomology $H_{p,n-1}(\Omega) = \frac{\{f \in L^2_{(p,n-1)}(\Omega) | \overline{\partial}f = 0\}}{\{f \in L^2_{(p,n-1)} | f = \overline{\partial}u, u \in L^2_{(p,n-2)}\}}$ is infinite dimensional.

- We use strictly plurisuperharmonic functions as weights near the pseudoconcave boundary.
- The $C^\infty(\overline{\Omega})$ result holds even for $\Omega_2$ with only Lipschitz boundary $b\Omega_2$ (Michel-Shaw 1999).
In recent years, there have been a lot of interests in studying the existence or nonexistence of Levi-flat hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n \geq 2$.

**Definition 1.** Let $M$ be a $C^2$ hypersurface in a complex manifold $\chi$. $M$ is called Levi-flat if the Levi-form vanishes on $M$.

A $C^2$ Levi-flat hypersurface is locally foliated by complex submanifolds of complex dimension $n - 1$. Locally $M = \bigcup_t \Sigma_t$ where each $\sigma_t$ is a complex manifold (leaf) and the foliation is of class $C^2$ in the transversal direction $t$ (Barrett-Fornaess).

**Definition 2.** Let $M$ be a Lipschitz (or $C^1$) hypersurface in a complex manifold $\chi$ (locally $M$ is the graph of a Lipschitz function). $M$ is called Levi-flat if locally $M$ foliated by complex submanifolds of complex dimension $n - 1$ such that the foliation is Lipschitz (or $C^1$) in the transversal direction.
Some known facts

- There are no compact Riemann surfaces or $C^2$ Levi-flat hypersurfaces in $\mathbb{C}^n$.
- There are compact Riemann surfaces in $\mathbb{C}P^n$.
- There are non-smooth (non-Lipschitz) Levi-flat hypersurfaces in $\mathbb{C}P^n$.

\[ M = \{ [z_0, z_1, z_2] \in \mathbb{C}P^2 \mid |z_1| = |z_2| \}, \]
\[ \mathbb{C}P^2 \setminus M = \Omega^+ \cup \Omega^-, \]

where $\Omega^+$ and $\Omega^-$ are pseudoconvex domains in $\mathbb{C}P^2$. The boundary is not a Lipschitz graph near the point $[1, 0, 0]$. Notice that $M$ is the union of compact Riemann surfaces, but not foliated at $[1, 0, 0]$. 
Recent results

- **Lins-Neto (1999)** There exist no real analytic Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$.

- **Siu (2000)** There exist no $C^\infty$ Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$.

- **Cao-Shaw (2007)** There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$.

- **Cao-Shaw-Wang (2004)** There exist no $C^2$ Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$.

There is still a gap in the proof for the case $n = 2$. 
Outline of the proof

- Following Siu’s idea, suppose that there exists a Lipschitz Levi-flat hypersurface $M$ in $\mathbb{C}P^n$, $n \geq 2$. Since $\mathbb{C}P^n$ has positive curvature, then curvature form $\Theta^N$ for the complex norma line bundle is positive (being a quotient bundle) when restricted to the leaf.

- If one can find a real continuous function $h$ on $M$ such that $\Theta^N = i\partial \overline{\partial} h$ on $M$, then $h$ reaches a maximum on some point $p$ in some leaf. This contradicts the maximum principle since $h$ is strictly pluri-subharmonic on the leaf containing $p$.

- $\Theta^N$ is $d$–exact. The problem is reduced to solve a $\overline{\partial}_b$ problem on $M$ with a continuous solution.
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Let $M$ be a compact hypersurface in $\mathbb{C}P^n$. The tangential Cauchy-Riemann equation $\bar{\partial}_b$:

$$\bar{\partial}_b u = f, \quad \text{where } f \text{ is a } (0,q)\text{-form on } M.$$

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(Universities of Notre Dame) The Cauchy-Riemann Equations in Complex f
The $\overline{\partial}$-equation on pseudoconvex domains in $\mathbb{CP}^n$:

Let $\mathbb{CP}^n$ be equipped with the standard Fubini-Study metric $\omega$.

- **Takeuchi** The (normalized) distance function $\delta$ from a point in $\Omega$ to the boundary $b\Omega$ satisfies the strong Oka’s condition:

  $$i\partial\overline{\partial}(-\log \delta) \geq \omega.$$

  This implies that $\Omega$ is Stein.

- Then for each $t > 0$, we can use $\phi_t = -t \log \delta$ to be the weight function in Hörmander’s theory and study the weighted $\overline{\partial}$-Neumann problem. However, $\phi$ is not continuous up to the boundary.
The $\overline{\partial}$-equation on pseudoconvex domains in $\mathbb{C}P^n$:

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Weighted $L^2$ theory

For $t > 0$, let $L^2(e^{-\phi_t}, \Omega) = L^2(\delta^t, \Omega) = L^2(\delta^t)$ be the weighted $L^2$ space with respect to the weight function $\phi_t = -t \log \delta$. The norm in $L^2(\delta^t)$ is denoted by $\| \|_{(t)}$. Let $\overline{\partial}$ and $\overline{\partial}^*$ be the closure of $\partial$ and its $L^2$ adjoint with respect to the weighted $L^2(\delta^t)$ space.

**Proposition 1** Let $\Omega \subset \subset \mathbb{C}P^n$ be a pseudoconvex domain. For any $t > 0$ and $(p, q)$-form $f \in L^2(\delta^t)$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, such that $\overline{\partial}f = 0$ in $\Omega$, there exists $u \in L^2(p, q-1)(\delta^t)$ satisfying $\overline{\partial}u = f$ and

$$\|u\|_{(t)}^2 \leq \frac{1}{t} \|f\|_{(t)}^2.$$

Furthermore, the weighted $\overline{\partial}$-Neumann operator $N_t$ exists for all $t > 0$.

How to get rid of the weight?
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How to get rid of the weight?
**Definition** A pseudoconvex domain is called hyperconvex if it has a bounded plurisubharmonic exhaustion function.

- **Diederich-Fornaess** Let $\Omega$ be a bounded $C^2$ pseudoconvex domain in $\mathbb{C}^n$. There exists $0 < t_0 \leq 1$ such that

$$i\partial\bar{\partial}(-\delta^{t_0}) \geq 0$$

where $\delta$ is some distance function of the form $\delta_0 e^{-t|z|^2}$ for the Euclidean distance function $\delta_0$ and $t \gg 1$ is a large constant.

- **Kerzman-Rosay, Damailly** For $C^1$ or Lipschitz domains, there exists bounded continuous plurisubharmonic exhaustion functions. In fact, one can have Hölder continuous bounded exhaustion function (Harrington 2007).
Ohsawa-Sibony Let $\Omega$ be a $C^2$ pseudoconvex domain in $\mathbb{C}P^n$. There exists a bounded plurisubharmonic exhaustion function for $\Omega$. There exists $0 < t_0 \leq 1$ such that

$$i\partial\overline{\partial}(-\delta^{t_0}) \geq 0$$

where $\delta$ is the distance function.

It is not known if one can have bounded plurisubharmonic function for Lipschitz or $C^1$ domains in $\mathbb{C}P^n$. 

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Lemma The following three conditions are equivalent:
There exists $0 < t_0 \leq 1$ and some distance function $\delta$ with

- $i\partial\overline{\partial}(-\log \delta) \geq t_0 i\frac{\partial\delta}{\overline{\partial}\delta} \overline{\partial}\delta$.
- $i\partial\overline{\partial}(-\delta^{t_0}) \geq 0$.
- For any $0 < t < t_0$, $i\partial\overline{\partial}(-\delta^t) \geq C_t \delta^t$.

Theorem Let $\Omega \subset \subset \mathbb{C}P^n$ be a pseudoconvex domain with $C^2$-smooth boundary $b\Omega$. Then $\square_{(p,q)}$ has closed range and the $\overline{\partial}$-Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \rightarrow L^2_{(p,q)}(\Omega)$ exists for every $p, q$ such that $0 \leq p \leq n, 0 \leq q \leq n$. Furthermore, there exists $0 < t_0 \leq 1$ such that the $\overline{\partial}$-Neumann operator $N, \overline{\partial}N, \overline{\partial}^* N$ and the Bergman projection $P$ are exactly regular on $W^s_{(p,q)}(\Omega)$ for $0 \leq s < \frac{1}{2}t_0$ with respect to the $W^s(\Omega)$-Sobolev norms.
Assume $\Omega^+$ is a pseudoconcave domain in $\mathbb{C}P^n$, $\Omega^+ = \mathbb{C}P^n \setminus \overline{\Omega}$.

For the $\overline{\partial}$-equation on a weakly pseudoconcave domain in $\mathbb{C}P^n$ with Lipschitz boundary, we have the following results.

**Theorem** Let $\Omega^+$ be a pseudoconcave domain in $\mathbb{C}P^n$ with Lipschitz boundary, where $n \geq 3$. For any $f \in W^{1+\epsilon}_{(p,q)}(\Omega^+)$, where $0 \leq p \leq n$, $1 \leq q < n - 1$, $p \neq q$ and $0 < \epsilon < \frac{1}{2}$, such that $\overline{\partial}f = 0$ in $\Omega^+$, there exists $u \in W^{1+\epsilon}_{(p,q-1)}(\Omega^+)$ with $\overline{\partial}u = f$ in $\Omega^+$. If $b\Omega^+$ is $C^2$, the statement is also true for $\epsilon = 0$. 

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The $\overline{\partial}$-Cauchy problem

The proof of the Theorem depends on the $\overline{\partial}$-closed extension of forms.

**$\overline{\partial}$-Cauchy problem** Let $\Omega \subset \subset \mathbb{C}P^n$ be a pseudoconvex domain with Lipschitz boundary, $n \geq 3$. Suppose that $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$ for some $t > 0$. We assume that

- if $1 \leq q \leq n - 1$, $\overline{\partial}f = 0$ in $\mathbb{C}P^n$ with $f = 0$ outside $\Omega$.
- if $q = n$, $\int_{\Omega} f \wedge h = 0$ for all $h \in L^2_{(n-p,0)}(\delta^t, \Omega)$.

Then there exists $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$ with $u_t = 0$ outside $\Omega$ satisfying $\overline{\partial}u_t = f$ in the distribution sense in $\mathbb{C}P^n$. If $b\Omega$ is $C^2$, the statement is also true for $t = 0$. 
The $\bar{\partial}$-Cauchy problem

Let $N_t$ be the weighted $\bar{\partial}$-Neumann operators $N_t$ for forms in $L^2_{(n-p,n-q)}(\delta^t, \Omega)$. Let $\star(t)$ denote the Hodge-star operator with respect to the weighted norm $L^2(\delta^t, \Omega)$, i.e., $\star(t) = \delta^t \star = \star \delta^t$.

Let $u_t$ be defined by

$$u_t = - \star(t) \bar{\partial} N_t \star(-t)f.$$

Then $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$ with $\bar{\partial} u_t = f$ if $q \leq n - 1$.

When $q = n$, we have $\bar{\partial} u_t = f - B_t(\star f) = f$ in $\Omega$.

Setting $u_t = 0$ outside $\Omega$, we check directly from integration by parts that $\bar{\partial} u_t = f$ in $\mathbb{C}P^n$.

Lipschitz boundary is necessary since we need that smooth forms are dense in $\text{Dom} (\bar{\partial})$ in the graph norm.
Let $M$ be a Lipschitz hypersurface in $\mathbb{C}P^n$ with $n \geq 3$.

- For any $f \in W^{1+\epsilon}_{(0,1)}(M)$ with $\overline{\partial}_b f = 0$, there exists a $u \in W^{1+\epsilon}(M)$ satisfying $\overline{\partial}_b u = f$.

- The $\overline{\partial}_b$ operator is the Cauchy-Riemann operator on the leaf, which is an elliptic operator. Let $(z', t)$ be local coordinates for $M$, then $\overline{\partial}_b u(z', t) = \overline{\partial}_{z'}(z', t)$.

- If we can have a trace (as a distribution) of the solution $u$ on each leaf, then the regularity for the elliptic theory can be applied to obtain the regularity on each leaf first.

- It remains to show that the solution is continuous in the transversal direction $t$. We introduce the Besov norms to show that $u$ is Hölder continuous if $f$ is.
Let $M$ be a Lipschitz hypersurface in $\mathbb{C}P^n$ with $n \geq 3$.

- For any $f \in W^{1/2+\epsilon}_{(0,1)}(M)$ with $\overline{\partial}_b f = 0$, there exists a $u \in W^{1+\epsilon}(M)$ satisfying $\overline{\partial}_b u = f$.

- The $\overline{\partial}_b$ operator is the Cauchy-Riemann operator on the leaf, which is an elliptic operator. Let $(z', t)$ be local coordinates for $M$, then $\overline{\partial}_b u(z', t) = \overline{\partial}_{z'}(z', t)$.

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- It remains to show that the solution is continuous in the transversal direction $t$. We introduce the Besov norms to show that $u$ is Hölder continuous if $f$ is.
(\(C^\infty\) regularity for \(\partialbar\)) Let \(\Omega \subset \subset \mathbb{C}P^n\), \(n \geq 2\), be a pseudoconvex domain with \(C^\infty\) boundary. For any \(f \in C^\infty(\Omega)\) with \(\partialbar f = 0\), there exists \(u \in C^\infty(\Omega)\) such that \(\partialbar u = f\).

- When \(\Omega\) is in \(\mathbb{C}^n\) or a Stein manifold, this is proved by Kohn.
- In fact, one would like to know if there exists a solution \(u \in W^1(\Omega)\). This will imply the nonexistence of Levi-flat hypersurfaces in \(\mathbb{C}P^2\).
(Strong Liouville’s Theorem) Let $\Omega^+ \subset \subset \mathbb{C}P^n$ be a pseudoconcave domain with $C^2$-smooth boundary (or Lipschitz) $b\Omega^+$, $n \geq 2$. Then $L^2_{(p,0)}(\Omega^+) \cap \text{Ker}(\partial) = \{0\}$ for every $1 \leq p \leq n$ and $L^2_{(0,0)}(\Omega^+) \cap \text{Ker}(\partial) = \mathbb{C}$.

- The set $W^{1+\epsilon}_{(p,0)}(\Omega^+) \cap \text{Ker}(\partial)$ is either zero or constants for Lipschitz pseudoconcave domains. When the boundary is $C^2$, this is also true for $\epsilon = 0$.

- This also will imply the nonexistence of Levi-flat hypersurfaces in $\mathbb{C}P^2$ since $L^2$ holomorphic functions or forms separate points on $C^2$ pseudoconvex domains.
Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}P^n$ with $C^2$ boundary, where $n \geq 2$. Then $H_{(p,0)}(\Omega) \cap W^1_{(p,0)}(\Omega)$ separates points for every $0 \leq p \leq n$.

- Notice that from Proposition 5.1, we have that $H(\Omega) = L^2(\Omega) \cap \text{Ker}(\bar{\partial})$ separates points, even on the Hartogs’ triangle $\Omega = \{[z_0, z_1, z_2] | |z_1| < |z_2|\} \subset \mathbb{C}P^2$.
- On the Hartogs’ triangle $\Omega$, we have that $H(\Omega) \cap W^1(\Omega) \neq \{\mathbb{C}\}$, but it does not separate points (joint work with Chakrabarti). Also $H(\Omega) \cap W^1(\Omega)$ is not dense in the Bergman space $H(\Omega)$.
- But $H(\Omega) \cap W^{1+\epsilon}(\Omega) = \{\mathbb{C}\}$ for any $\epsilon > 0$. 
Let \( \Omega \) be a \( C^2 \) pseudoconcave domain in \( \mathbb{C}P^n \), \( n \geq 2 \). For any \( 0 \leq p \leq n \) and \( 1 \leq q \leq n - 1 \), the range of \( \overline{\partial} : L^2_{(p,q-1)}(\Omega) \rightarrow L^2_{(p,q)}(\Omega) \) is closed. The \( \overline{\partial} \)-Neumann operator \( N_{(p,q)} \) exists on \( L^2_{(p,q)}(\Omega) \). Furthermore, we have for \( 1 \leq q < n - 2 \), \( \square_{(p,q)} N_{(p,q)} = I \) on \( L^2_{(p,q)}(\Omega) \).
Let $\Omega$ be a $C^2$ pseudoconcave domain in $\mathbb{C}P^n$, $n \geq 2$. For any $0 \leq p \leq n$, the space of harmonic $(p, n - 1)$-forms $H_{(p,n-1)}$ is infinite dimensional and for any $f \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap (H_{(p,n-1)})^\perp$, we have

$$\|f\|^2 \leq C(\|\bar{\partial}f_1\|^2 + \|\bar{\partial}^*f_2\|^2).$$

Notice that when the boundary is strictly pseudoconcave, Question 4 are known.
Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}P^n$ with $C^2$ boundary, where $n \geq 2$. Then the range of $\overline{\partial}_b : L^2_{(p,q-1)}(b\Omega) \to L^2_{(p,q)}(b\Omega)$ is closed in the $L^2_{(p,q)}(b\Omega)$ space for all $0 \leq p \leq n$ and $1 \leq q \leq n - 1$.

When $\Omega$ is a smooth pseudoconvex in $\mathbb{C}^n$, this is proved. If $\Omega$ is Lipschitz pseudoconvex in $\mathbb{C}^n$ and we assume that there exists a plurisubharmonic defining function in a neighborhood, it is also true.
Thank you!