Vortices in Non-Abelian Gauge Field Theory

Yisong Yang

Polytechnic Institute of New York University

Talk at
Nonlinear Phenomena
A View From Mathematics and Physics
National Taiwan University, Taipei, January 10 – 14, 2011

(Joint work with Chang-Shou Lin)

Outline

1. Big picture: from monopole to color confinement
2. Abelian vortices
3. Vortices in non-Abelian gauge theory
4. Non-Abelian vortices: recent development
5. Non-Abelian vortices: existence theorems
6. Remarks on methods and applications

Phenomena without Confinement

Gravitational force

Electro-static force

Magneto-static force

$$F = \mathrm{O}(r^{-2}). \quad \text{(Newton–Coulomb type law)}$$

Energy needed to separate two interacting particles is finite,

$$E = \int_R^\infty F\, dr < \infty.$$

Phenomena with Confinement

Hook's elastic force

$$F = kr.$$

Hypothetical inter-quark color force

$$F = k.$$

Both models result in infinite separation energy

$$E = \int_R^\infty F\,dr = \infty.$$

In other words, it is impossible to separate two interacting particles. Such a phenomenon is called "confinement."

Linear confinement

In the case when the inter-particle force is constant,

$$F = k,$$

the energy needed to separate two particles to a distance r apart is

$$E = \int F\, dr = kr + E_0,$$

which is a linear function of r.
Such a confinement model is called "linear confinement" which will be our concern here.

Monopole confinement

In a normal situation, a magnetic monopole and anti-monopole can be separated due to the $O(r^{-2})$ magnetic force decay law provided by the Maxwell equations as mentioned earlier.

In a type-II superconductor, however, they will follow a linear confinement law and thus cannot be separated.

How?

Types of Superconductivity versus the Meissner effect

Type-I superconductors: complete screening of magnetic field (no penetration of magnetic field)

Type-II superconductors: partial screening of magnetic field (partial penetration of magnetic field in the form of thin vortex tubes)

The latter phenomenon is important because it provides a mechanism under which the magnetic force in the form of vortex lines does not decay with respect to space separation.

Monopole confinement achieved

In other words, if a monopole and anti-monopole are placed inside a type-II superconductor, the attractive magnetic force between these monopoles is established through thin vortex tubes so that the force maintains constant regardless of the distance between the monopoles.

Conclusion: Linear confinement for monopoles

Lesson: Vortices are not only a phenomenon in superconductors, but also useful for a fundamental mechanism.

Quarks and elementary particles

A quark is an elementary particle and a fundamental constituent of matter. Quarks combine to form composite particles called hadrons.

A hadron is a composite particle made of quarks held together by the strong (color) force. Hadrons are categorized into two families: baryons (made of three quarks), and mesons (made of one quark and one anti-quark).

The best-known hadrons are protons and neutrons (both baryons)

Color (or quark) confinement (cf. Wikipedia)

Color confinement, often simply called confinement, is the physical phenomenon that color charged particles (such as quarks) cannot be isolated singularly, and therefore cannot be directly observed. Quarks, by default, clump together to form groups, or hadrons. The constituent quarks in a group cannot be separated from their parent hadron, and this is why quarks can never be studied or observed in any more direct way than at a hadron level.

The reasons for quark confinement are somewhat complicated. No analytic proof exists that QCD should be confining. As two quarks separate, the gluon fields form narrow tubes (or strings) of color charge, which tend to bring the quarks together. The color force experienced by the quarks in the direction to hold them together, remains constant, regardless of their distance from each other.

Analogy between the monopole and quark confinements

The afore-described analogy between monopole and quark confinements was formulated independently and almost simultaneously by Nambu, 't Hooft, Mandelstam, in 1970s.

Ingredients of color (quark) confinement

(Color-charged) non-Abelian monopoles = quarks

non-Abelian chromoelectric flux tubes = non-Abelian vortex lines

Operation through "non-Abelian dual Meissner effect"

non-Abelian monopoles: lot of work

non-Abelian vortices: an area of research of contemporary interest

Vortices in Abelian Higgs model

The energy density in 2D is

$$\mathcal{H} = \frac{1}{4}F_{jk}^2 + \frac{1}{2}|D_j\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2, \quad j,k = 1,2,$$

where

$$F_{jk} = \partial_j A_k - \partial_k A_j, \quad D_j\phi = \partial_j\phi - \mathrm{i}A_j\phi, \quad \lambda > 0.$$

The equations of motion (or the Ginzburg–Landau equations for superconductivity) are given by

$$D_j D_j\phi = \frac{\lambda}{2}(|\phi|^2 - 1)\phi, \quad \partial_j F_{jk} = \frac{\mathrm{i}}{2}(\overline{\phi}D_k\phi - \phi\overline{D_k\phi}).$$

Mutiple vortices in BPS equations

In the BPS limit $\lambda = 1$, the equations are reduced to

$$D_1\phi + \mathrm{i}D_2\phi = 0,$$

$$F_{12} + \frac{1}{2}(|\phi|^2 - 1) = 0.$$

It is seen clearly that the maxima of the vorticity or magnetic field F_{12} are attained at the zeros of the Higgs field ϕ (this picture is known as the "Higgs-condensed vortices").

"The largest penetration of magnetic field is at the spots where superconductivity is destroyed" (this picture is exactly the Meissner effect described earlier).

Existence and uniqueness theorem

Multiple Vortex Problem. For any

$$Z = \{p_1, \cdots, p_n\},$$

construct a solution realizing Z as the set of zeros of the Higgs field ϕ, known as an *n*-vortex solution.

Theorem. (i) For any Z, there is a unique solution over $\mathbf{R}^2$ (Taubes).
(ii) For any Z contained in a doubly periodic domain Ω, there is a unique solution over Ω (the Abrikosov condensate) if and only if

$$4\pi n < |\Omega|.$$

(Wang – Yang)

The governing PDE is

$$\Delta u = e^u - 1 + 4\pi \sum_{i=1}^{n} \delta_{p_i}(x)$$

This PDE is of the Liouville type which is not integrable.

Note: The equation

$$\Delta u = e^u - \varepsilon$$

is integrable if and only if $\varepsilon = 0$.

Topological reason for the existence of an n-vortex solution over $\mathbf{R}^2$

Finite energy implies $|\phi| \to 1$ as $|x| \to \infty$.
So ϕ maps the circle S^1 at infinity into S^1 which is the manifold of the gauge group $U(1)$. ("The vacuum manifold")
Since $\pi_1(U(1)) = \pi(S^1) = \mathbf{Z}$, we see that there is an abundance of solutions labeled by integers in $\mathbf{Z}$, or the n-vortex solutions.

Multiple vortex solutions generated from nontrivial windings of the Higgs field ϕ.

"The Higgs condensed vortices exist."

Vortices in non-Abelian gauge theory

Take the typical gauge group $SU(N)$.
It is well known that $\pi_1(SU(N)) = 0$ (trivial).

"The Higgs condensed vortices do not exist in non-Abelian gauge field theory" (?)

Example: The electroweak theory of Weinberg–Salam

The gauge group is $SU(2) \times U(1)$.

In the adjoint representation of the Higgs field ϕ, the vacuum manifold $\mathcal{M}$ is $SU(2)$.

In the fundamental representation of the Higgs field, $\mathcal{M}$ is the unit sphere in $\mathbf{C}^2$ which is S^3.

In either case,

$$\pi_1(\mathcal{M}) = 0.$$

"No Higgs condensed vortices in the electroweak theory."

The W-condensed electroweak vortices

Surprisingly, in 1989, Ambjorn and Olesen showed that electroweak vortices should exist in the unitary gauge which are generated by the W-bosons (a mediating particle responsible for weak interaction). In their work, the multiple vortex BPS equations governing the interaction of photons, W- and Z-bosons, and the Higgs particles, are derived to be

$$
\begin{aligned}
D_1 W + \mathrm{i} D_2 W &= 0, \\
P_{12} &= \frac{g}{2\sin\theta}\varphi_0^2 + 2g\sin\theta|W|^2, \\
Z_{12} &= \frac{g}{2\cos\theta}(\varphi^2 - \varphi_0^2) + 2g\cos\theta|W|^2, \\
Z_j &= -\frac{2\cos\theta}{g}\epsilon_{jk}\partial_k \ln\varphi, \quad j,k = 1,2.
\end{aligned}
$$

Here W is a complex scalar field, P_j is the photon field, Z_j is the Z-vector field, φ is a real scalar field representing the Higgs doublet which never vanishes,

$$D_j W = \partial_j W - \mathrm{i}g(P_j \sin\theta + Z_j \cos\theta)W,$$

$g > 0$ is a coupling constant, $\varphi_0 > 0$ is the expectation value of ϕ, θ is the Weinberg mixing angle, and

$$P_{12} = \partial_1 P_2 - \partial_2 P_1, \quad Z_{12} = \partial_1 Z_2 - \partial_2 Z_1,$$

are the P- and Z-field strengths (curvatures).

Observation: The minima of P_{12} are attained at the zeros of W.

"Anti-screening of the magnetic field" or "an anti-Meissner effect"

Multiple vortex problem

Over $\mathbf{R}^2$, any nontrivial solution carries infinite energy.

Over a doubly periodic domain Ω, we are to look for solutions so that W vanishes exactly at the prescribed set of zeros

$$Z = \{p_1, \cdots, p_n\}.$$

The associated PDE problem

We are to solve the system of nonlinear elliptic PDEs

$$\begin{aligned} \Delta u &= -4g^2 e^u - g^2 e^w + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \\ \Delta w &= 2g^2 e^u + \frac{g^2}{2\cos^2\theta}(e^w - \varphi_0^2). \end{aligned}$$

Basic tool as well as limitation: The Moser–Trudinger inequality

$$\int_\Omega e^f \, dx \leq C_0 \exp\left(\frac{1}{16\pi}\int_\Omega |\nabla f|^2 \, dx\right), \quad \int_\Omega f \, dx = 0.$$

Existence theory for W-condensed vortices

Theorem (Spruck – Yang) A necessary condition condition for the existence of an n-vortex solution is

$$g^2\varphi_0^2 < \frac{4\pi n}{|\Omega|} < \frac{g^2\varphi_0^2}{\cos^2\theta}. \tag{1}$$

In addition, if

$$\frac{4\pi n}{|\Omega|} < \frac{8\pi\sin^2\theta}{|\Omega|} + g^2\varphi_0^2, \tag{2}$$

then a solution exists. In particular, when $n = 1, 2$, the condition (2) is contained in the condition (1) so that (1) is necessary and sufficient for the existence of a solution when $n = 1, 2$.

Improved result of Bartolucci and Tarantello

In the afore-stated theorem, the sufficiency condition (2) for the existence of an n-vortex solution is

$$\frac{4\pi n - g^2\varphi_0^2|\Omega|}{8\pi\sin^2\theta} < 1.$$

Through a blow-up study based on a Liouville type equation with singular source terms, Bartolucci and Tarantello were able to obtain the improved sufficiency condition

$$\frac{4\pi n - g^2\varphi_0^2|\Omega|}{8\pi\sin^2\theta} < 2, \quad \frac{4\pi n - g^2\varphi_0^2|\Omega|}{8\pi\sin^2\theta} \neq 1.$$

In particular, (1) is necessary and sufficient for $n = 1, 2, 3, 4$ provided $n \neq \frac{g^2\varphi_0^2}{4\pi}|\Omega| + 2\sin^2\theta$.

Non-Abelian vortices: recent development

Marshakov and Yung,
Hanany and Tong
Auzzi, Bolognesi, Evslin, Konishi, and Yung
Others

Verdict
There is an abundance of the Higgs condensed vortices in non-Abelian theory.

How?
If the gauge group is G so that the symmetry of the model is spontaneously broken, with an unbroken (isotropy) subgroup H, then the vacuum manifold is not G but the quotient space G/H, whose fundamental group may not be trivial.

A concrete but important example

Take a $G = SU(N) \times U(1)$ gauge field theory.
The unbroken subgroup may assume a discrete form, $H = \mathbf{Z}_N$.

The vacuum manifold is then

$$\frac{SU(N) \times U(1)}{\mathbf{Z}_N} \approx U(N).$$

However, we have the classical result

$$\pi_1(U(N)) = \mathbf{Z}.$$

So the abundance of topologically nontrivial vortex solutions follows.

Unified (mixed) monopole-vortex equations in $\mathbf{R}^3$

In the context of an $\mathcal{N} = 2$ supersymmetric field-theoretical formalism containing a scalar field ϕ in the adjoint representation of $U(N)$ and $N = N_{\text{flavor}}$ flavor scalar fields q_f $(f = 1, \cdots, N)$ in the fundamental representation of $SU(N)$, Tong obtained the unified monopole-vortex equations governing the interaction (or confinement) of non-Abelian monopoles (quarks) through vortex tubes (color forces)

$$B_1 = D_1\phi, \quad B_2 = D_2\phi, \quad B_3 = D_3\phi + e^2\left(\sum_{f=1}^{N} q_f q_f^\dagger - v_0^2\right),$$

$$D_1 q_f = \mathrm{i} D_2 q_f, \quad D_3 q_f = -(\phi - m_f) q_f, \quad f = 1, \cdots, N.$$

Existence of non-Abelian vortices

Recent work with Chang-Shou Lin on the existence theory of the Higgs condensed multiple BPS vortices in non-Abelian gauge field theory has been reported in two joint papers of ours:

1. Non-Abelian Multiple Vortices in Supersymmetric Field Theory (to appear in CMP).

2. Sharp Existence and Uniqueness Theorems for Non-Abelian Multiple Vortex Solutions (preprint).

The results of this line of work will be described below.

Vortices in supersymmetric gauge theory

(Following Eto, Fujimori, Nagashima, Nitta, Ohashi, Sakai)

Let W_μ and w_μ be gauge fields given over the groups $U(N)$ and $U(1)$ respectively.
Taking the fundamental representation, the N Higgs fields are given by an N by N complex matrix H.
The Lagrangian density is defined by $\mathcal{L} = \mathcal{K} - \mathcal{V}$, with

$$\begin{aligned} \mathcal{K} &= \mathrm{Tr}\left(-\frac{1}{2g^2}(F_{\mu\nu})^2 + \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger \right) - \frac{1}{4e^2}(f_{\mu\nu})^2, \\ \mathcal{V} &= \frac{g^2}{4}\mathrm{Tr}(\langle HH^\dagger\rangle^2) + \frac{e^2}{2}(\mathrm{Tr}(HH^\dagger - c\mathbf{1}_N))^2, \end{aligned}$$

where $\langle X\rangle = X - \mathrm{Tr}(X)\mathbf{1}_N$ stands for the traceless part of an $N\times N$ matrix X,

$$\mathcal{D}_\mu H = \partial_\mu H + \mathrm{i}W_\mu H + \mathrm{i}w_\mu H,$$

the non-Abelian and Abelian gauge field curvatures are given by

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + \mathrm{i}[W_\mu, W_\nu], \quad f_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu,$$

and $g, e, c > 0$ are coupling parameters.
The non-Abelian and Abelian gauge fields are both massive with the mass spectra

$$m_g = g\sqrt{c}, \quad m_e = e\sqrt{2Nc}.$$

BPS system of equations

The equations of motion are hard to solve.

Suppose the fields depend on x^1 and x^2 only.
There hold the BPS equations

$$
\begin{aligned}
\mathcal{D}_1 H + \mathrm{i}\mathcal{D}_2 H &= 0, \\
F_{12} &= \frac{m_g^2}{2c}\langle HH^\dagger\rangle, \\
f_{12} &= \frac{m_e^2}{2c}\mathrm{Tr}(HH^\dagger - c\mathbf{1}_N).
\end{aligned}
$$

PDEs

In terms of two real-valued functions ψ_e and ψ_g, the BPS system away from the vortex points becomes

$$\begin{aligned}\Delta\psi_e + \frac{m_e^2}{N}\mathrm{e}^{-\psi_e}(\mathrm{e}^{-\psi_g} + [N-1]\mathrm{e}^{\frac{\psi_g}{N-1}}) &= m_e^2,\\ \Delta\psi_g + \frac{m_g^2(N-1)}{N}\mathrm{e}^{-\psi_e}(\mathrm{e}^{-\psi_g} - \mathrm{e}^{\frac{\psi_g}{N-1}}) &= 0.\end{aligned}$$

If we prescribe point vortices at $p_1, \cdots, p_n \in \Omega$, then

$$N\psi_e + \ln|x-p_j|^2 \quad \text{and} \quad \frac{N}{N-1}\psi_g + \ln|x-p_j|^2$$

are smooth functions for $x \in \mathbf{R}^2$ near p_j, $\quad j = 1, \cdots, n$.

PDEs (continued)

When we set

$$-N\psi_e = u_1, \quad -\frac{N}{N-1}\psi_g = u_2,$$

we arrive at the governing equations:

$$\begin{aligned}\Delta u_1 &= -Nm_e^2 + m_e^2\left(\mathrm{e}^{\frac{u_1}{N}+\frac{(N-1)}{N}u_2} + [N-1]\mathrm{e}^{\frac{u_1}{N}-\frac{u_2}{N}}\right) + 4\pi\sum_{j=1}^{n}\delta_{p_j}(x),\\ \Delta u_2 &= m_g^2\left(\mathrm{e}^{\frac{u_1}{N}+\frac{(N-1)}{N}u_2} - \mathrm{e}^{\frac{u_1}{N}-\frac{u_2}{N}}\right) + 4\pi\sum_{j=1}^{n}\delta_{p_j}(x).\end{aligned}$$

Over $\mathbf{R}^2$, the solution is subject to the boundary condition

$$u_1 \to 0, \quad u_2 \to 0 \quad \text{as } |x| \to \infty.$$

Existence and uniqueness results

Theorem 1

(i) Over $\mathbf{R}^2$ there is always a unique solution.

(ii) Over a doubly periodic domain Ω, for any n, there is an n-vortex solution if and only if

$$|\Omega| > \frac{4\pi n}{N m_e^2} + \frac{4\pi n(N-1)}{N m_g^2}.$$

Moreover, when a solution exists, it must be unique.

Vortices in non-Abelian gauge theory

(Following Gudnason, Jiang, and Konishi)

The Lagrangian density is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4e^2}F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + (\mathcal{D}_\mu q_f)^\dagger \mathcal{D}^\mu q_f \\ &\quad -\frac{e^2}{2}\left|q_f^\dagger t^0 q_f - \frac{v_0^2}{\sqrt{4N}}\right|^2 - \frac{g^2}{2}\left|q_f^\dagger t^a q_f\right|^2,\end{aligned}$$

for which the gauge group G is of the general form $G = G' \times U(1)$ where G' is typically chosen to be $G' = SO(2N)$ or $G' = USp(2N)$ (the unitary symplectic group).

Here, $\mu, \nu = 0, 1, 2, 3$ are spacetime indices, $a = 1, \cdots, \dim(G')$ labels the generators of G', the index 0 indicates the Abelian (U(1)) gauge field, $f = 1, \cdots, N_{\text{flavor}}$ labels the matter flavors or "scalar quark" fields, q_f, all are assumed to lie in the fundamental representation of G'. Besides, $A_\mu = A_\mu^0 + A_\mu^a t^a$,

$$\mathcal{D}_\mu q_f = \partial_\mu q_f + \mathrm{i} A_\mu q_f, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu],$$

where the generators of G' and $U(1)$, i.e., $\{t^a\}$ and t^0, are normalized to satisfy

$$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \quad t^0 = \frac{1}{\sqrt{4N}}\mathbf{1}_{2N}.$$

BPS system of equations

When the number of matter flavors is $N_{\text{flavor}} = 2N$, the scalar quark fields may be represented as a color-flavor mixed matrix q of size $2N \times 2N$. Assume that the fields depend only on x^1 and x^2. We arrive at the BPS system of equations

$$\begin{aligned} \mathcal{D}_1 q + \mathrm{i}\mathcal{D}_2 q &= 0, \\ F_{12}^0 - \frac{e^2}{\sqrt{4N}}(\mathrm{Tr}(qq^\dagger) - v_0^2) &= 0, \\ F_{12}^a t^a - \frac{g^2}{4}(qq^\dagger - J^\dagger (qq^\dagger)^T J) &= 0, \end{aligned}$$

where J is the standard symplectic matrix $J = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}$.

Further reduction

When G' contains $U(1)^N$, a further ansatz reduces the above BPS system into

$$\begin{aligned}
(\partial_1 + \mathrm{i}\partial_2)\phi &= \mathrm{i}([a_1 + \mathrm{i}a_2] + [b_1 + \mathrm{i}b_2])\phi, \\
(\partial_1 + \mathrm{i}\partial_2)\psi &= \mathrm{i}([a_1 + \mathrm{i}a_2] - [b_1 + \mathrm{i}b_2])\psi, \\
a_{12} = \partial_1 a_2 - \partial_2 a_1 &= -\frac{\alpha}{4}(|\phi|^2 + |\psi|^2 - \gamma), \\
b_{12} = \partial_1 b_2 - \partial_2 b_1 &= -\frac{\beta}{4}(|\phi|^2 - |\psi|^2),
\end{aligned}$$

where the positive parameters α, β, γ are given by

$$\alpha = e^2, \quad \beta = g^2, \quad \gamma = \frac{v_0^2}{N}.$$

Here ϕ, ψ are complex scalar fields, and a_j, b_j are real-valued vector fields.
An n-vortex solution is such that ϕ vanishes exactly at

$$Z = \{p_1, \cdots, p_n\},$$

but ψ never vanishes.

Over $\mathbf{R}^2$, the scalar fields ϕ and ψ are subject to the boundary condition

$$|\phi|, |\psi| \to \sqrt{\frac{\gamma}{2}} \quad \text{as } |x| \to \infty.$$

PDE problem

Now, setting $u = \ln|\phi|^2, v = \ln|\psi|^2$, we arrive at the coupled system

$$\begin{aligned}\Delta(u+v) &= \alpha(e^u + e^v - \gamma) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x),\\ \Delta(u-v) &= \beta(e^u - e^v) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x).\end{aligned}$$

Existence and uniqueness results

Theorem 2

(i) For any α, β, γ, the equations over $\mathbf{R}^2$ have a unique *n*-vortex solution.

(ii) For the equations over a doubly periodic domain Ω, there is an *n*-vortex solution if and only if

$$4\pi n\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) < \gamma|\Omega|.$$

The solution, if exists, must be unique which may always be constructed via a constrained minimization procedure.

An extended situation

(Again following Gudnason, Jiang, and Konishi)

In the case where $G' = SO(2N)$, the BPS system may be reduced into ($\overline{\partial} = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), a = a_1 + \mathrm{i}a_2, \Phi, \phi, \psi$ are all complex scalars)

$$\begin{aligned}
\overline{\partial}\Phi &= \mathrm{i}a\Phi, \\
\overline{\partial}\phi &= \mathrm{i}(a+b)\phi, \\
\overline{\partial}\psi &= \mathrm{i}(a-b)\psi, \\
a_{12} &= \frac{e^2}{4N}(v_0^2 - 2(N-1)|\Phi|^2 - |\phi|^2 - |\psi|^2), \\
b_{12} &= \frac{g^2}{4}(|\psi|^2 - |\phi|^2).
\end{aligned}$$

Relation between Φ, ϕ, ψ

We consider a situation where ψ has no zero but the zeros of Φ and ϕ coincide.

From the BPS equations, it can be established that

$$|\Phi|^4 = |\phi|^2|\psi|^2.$$

Thus, the terms and equations involving Φ can all be suppressed.

PDEs

Setting

$$u = \ln|\phi|^2, \quad v = \ln|\psi|^2, \quad \alpha = \frac{2e^2}{N}, \quad \beta = 2g^2, \quad \gamma = v_0^2,$$

we see that the BPS system becomes

$$\begin{aligned} \Delta(u+v) &= \alpha\left(2(N-1)e^{\frac{1}{2}(u+v)} + e^u + e^v - \gamma\right) + 4\pi\sum_{s=1}^{n}\delta_{p_s}(x), \\ \Delta(u-v) &= \beta\left(e^u - e^v\right) + 4\pi\sum_{s=1}^{n}\delta_{p_s}(x), \end{aligned}$$

Existence and uniqueness results

Theorem 3

(i) For any α, β, γ, the equations over $\mathbf{R}^2$ have a unique *n*-vortex solution.

(ii) For the equations over a doubly periodic domain Ω, there is an *n*-vortex solution if and only if

$$4\pi n \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < \gamma |\Omega|.$$

The solution, if exists, must be unique, which may be constructed via a constrained minimization procedure provided that $n = 1, 2, 3$ and $3\alpha < \beta$; or $n = 1, 2$ and $\alpha < \beta$; or $n = 1$ and $\alpha < 3\beta$.

When vortices are generated from both ϕ and ψ

(Following the earlier studies of
Auzzi, Bolognesi, Evslin, Konishi, and Yung
Marshakov and Yung
Shifman and Yung)

We now consider the case when both the complex-valued scalar fields ϕ and ψ are allowed to independently generate vortices with their respectively prescribed zero sets

$$Z_\phi = \{p_1, \cdots, p_m\}, \quad Z_\psi = \{q_1, \cdots, q_n\}.$$

We may call so-generated vortices "(m, n)-vortices" (?)

PDEs

The (m, n)-vortices are governed by the PDEs

$$\begin{aligned}\Delta(u+v) &= \alpha(e^u + e^v - \gamma) + 4\pi\sum_{s=1}^{m}\delta_{p_s}(x) + 4\pi\sum_{s=1}^{n}\delta_{q_s}(x),\\ \Delta(u-v) &= \beta(e^u - e^v) + 4\pi\sum_{s=1}^{m}\delta_{p_s}(x) - 4\pi\sum_{s=1}^{n}\delta_{q_s}(x),\end{aligned}$$

where

$$u = \ln|\phi|^2, \quad v = \ln|\psi|^2.$$

Existence and uniqueness of (m, n)-vortices

Theorem 4

(i) For any α, β, γ and m, n, the equations over $\mathbf{R}^2$ have a unique (m, n)-vortex solution.

(ii) For the equations over a doubly periodic domain Ω, there is an (m, n)-vortex solution if and only if m and n satisfy

$$4\pi \left(\frac{(m+n)}{\alpha} + \frac{|m-n|}{\beta} \right) < \gamma |\Omega|.$$

The solution, if exists, must be unique and can be constructed via a constrained minimization procedure.

Remarks on methods and applications

Calculus of variations

Degree theorem argument

Monotone iterations

For example, we consider the simplest situation where the governing equations are

$$\begin{aligned}\Delta(u+v) &= \alpha(e^u + e^v - 2),\\ \Delta(u-v) &= \beta(e^u - e^v),\end{aligned}$$

where we have ignored the source terms.

Taking $f = u + v, g = u - v$, we change system into

$$\begin{aligned}\Delta f &= \alpha(e^{\frac{1}{2}(f+g)} + e^{\frac{1}{2}(f-g)} - 2),\\ \Delta g &= \beta(e^{\frac{1}{2}(f+g)} - e^{\frac{1}{2}(f-g)}).\end{aligned}$$

Variational principle

It can be seen that the system is governed by the action functional

$$I(f,g) = \int \mathrm{d}x \left\{ \frac{1}{2\alpha}|\nabla f|^2 + \frac{1}{2\beta}|\nabla g|^2 + 2\left(\mathrm{e}^{\frac{1}{2}(f+g)} - 1 - \frac{1}{2}(f+g)\right) + 2\left(\mathrm{e}^{\frac{1}{2}(f-g)} - 1 - \frac{1}{2}(f-g)\right)\right\}.$$

Uniqueness follows from the convexity of this functional.

From $|\nabla(f+g)|^2 + |\nabla(f-g)|^2 = 2|\nabla f|^2 + 2|\nabla g|^2$, we may obtain suitable coerciveness for the functional.
Hence existence follows as well.

The Φ problem

We have seen that in the situation where $G' = SO(2N)$, the presence of the Φ-term leads us to the equations

$$\begin{aligned}
\Delta(u+v) &= \alpha\left(2(N-1)e^{\frac{1}{2}(u+v)} + e^u + e^v - \gamma\right) + 4\pi\sum_{s=1}^{n}\delta_{p_s}(x), \\
\Delta(u-v) &= \beta\,(e^u - e^v) + 4\pi\sum_{s=1}^{n}\delta_{p_s}(x).
\end{aligned}$$

For the full plane problem, the additional term $e^{\frac{1}{2}(u+v)}$ causes no trouble.
However, for the doubly periodic problem, some trouble arises due to the integral constraints we need to preserve.

The Φ problem continued

More precisely, we encounter a radical root problem which prevents us from establishing the existence without assuming additional conditions.

To overcome such a difficulty, we consider the modified equations

$$\begin{aligned}\Delta(u+v) &= \alpha\left(2(N-1)t\mathrm{e}^{\frac{1}{2}(u+v)}+\mathrm{e}^u+\mathrm{e}^v-\gamma\right)+4\pi\sum_{s=1}^n\delta_{p_s}(x),\\ \Delta(u-v) &= \beta\left(\mathrm{e}^u-\mathrm{e}^v\right)+4\pi\sum_{s=1}^n\delta_{p_s}(x),\quad 0\le t\le 1.\end{aligned}$$

We can get suitable *a priori* estimates to achieve "$\deg(P|_{t=1})=\deg(P|_{t=0})=1$" which allows us to "flow away" the troublesome term.

The condition for the existence of doubly-periodic (m, n)-vortices

The governing equations are

$$\begin{aligned} \Delta(u+v) &= \alpha(e^u+e^v-\gamma)+4\pi\sum_{s=1}^m \delta_{p_s}(x)+4\pi\sum_{s=1}^n \delta_{q_s}(x), \\ \Delta(u-v) &= \beta(e^u-e^v)+4\pi\sum_{s=1}^m \delta_{p_s}(x)-4\pi\sum_{s=1}^n \delta_{q_s}(x). \end{aligned}$$

Integrating these equations formally, we arrive at

$$\begin{aligned} 4\pi m\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-4\pi n\left(\frac{1}{\alpha}-\frac{1}{\beta}\right) &< \gamma|\Omega|, \\ 4\pi m\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)-4\pi n\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) &< \gamma|\Omega|. \end{aligned}$$

The above two inequalities may be combined into one to give us

$$4\pi\left(\frac{(m+n)}{\alpha}+\frac{|m-n|}{\beta}\right)<\gamma|\Omega|.$$

Another interesting equivalent form of the above inequality reads

$$4\pi\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)(m+n)<\gamma|\Omega|+\frac{8\pi}{\beta}\min\{m,n\}.$$

We then show that such a condition is sufficient for the existence of a solution. We can do so by a multiply-constrained minimization procedure.

Dimensionality count

As an application, we may comment on applying our results to dimensionality count for the moduli spaces of the BPS vortices.

For the classical Abelian Higgs model, E. Weinberg showed by using index theory that the moduli space $\mathcal{M}_n$ of n-vortices has the dimension

$$\dim(\mathcal{M}_n) = 2n.$$

The existence and uniqueness results of Taubes gives a concrete realization of such a count since the solutions are shown to be determined by the $2n$ coordinates of the n prescribed vortex points, as the zeros of the Higgs field.

Dimensionality count (continued)

For the non-Abelian BPS vortices, there have been some studies extending the calculation of Weinberg.

For example, Hanany and Tong

Our results are concrete realizations of the dimensionality counts for the moduli spaces of the BPS non-Abelian n-vortices and (m, n)-vortices, say $\mathcal{M}_n$ and $\mathcal{M}_{m,n}$, with

$$\begin{aligned} \dim(\mathcal{M}_n) &= 2n, \\ \dim(\mathcal{M}_{m,n}) &= 2(m+n), \end{aligned}$$

respectively.