Singular Limits for Integrable Equations

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Dispersive Linear Waves

Prototype: free Schrödinger equation.

Consider the free-particle (linear) Schrödinger equation

$$i\epsilon\frac{\partial\psi}{\partial t}+\frac{1}{2}\epsilon^2\frac{\partial^2\psi}{\partial x^2}=0,\quad x\in\mathbb{R},\quad t>0,\quad \epsilon>0,$$

with initial data

$$\psi(x,0)=\psi_0(x)=\sqrt{\rho_0(x)}e^{iS(x)/\epsilon},\quad S(x):=\int_0^x u_0(y)\,dy.$$

Overall goal: describe how ψ depends on

- the independent variables x and t
- the parameter ϵ (Planck's constant)
- the initial data ψ_0 (equivalently, the amplitude $\rho_0>0$ and phase gradient u_0).

The initial-value problem can be solved by the Fourier/Inverse-Fourier transform pair.

1. Direct transform: $\hat{\psi}_0(\lambda) := \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} \psi_0(x) e^{2i\lambda x/\epsilon}\,dx$.
2. Time evolution: $\hat{\psi}(\lambda, t) = e^{-2i\lambda^2 t/\epsilon}\hat{\psi}_0(\lambda)$.
3. Inverse transform: $\psi(x,t) = \dfrac{2}{\epsilon}\displaystyle\int_{\mathbb{R}} \hat{\psi}(\lambda, t) e^{-2i\lambda x/\epsilon}\,d\lambda$.

Combining steps 2 and 3 gives an integral representation of $\psi(x,t)$ in terms of the transform $\hat{\psi}_0(\lambda)$:

$$\psi(x,t) = \frac{2}{\epsilon}\int_{\mathbb{R}} \hat{\psi}_0(\lambda) e^{-2i(\lambda x + \lambda^2 t)/\epsilon}\,d\lambda.$$

Also using step 1 gives an iterated double integral representation of $\psi(x,t)$ in terms of $\psi_0(x)$ directly:

$$\psi(x,t) = \frac{1}{\pi\epsilon}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi_0(y) e^{-2i(\lambda(x-y) + \lambda^2 t)/\epsilon}\,dy\,d\lambda.$$

For some special $\psi_0(x)$ we might be able to evaluate the integrals exactly, but this is unusual.

But integrals can also be calculated with explicit leading-order terms in various singular limits. Typical example: $t \to \infty$ (long-time limit). It is more interesting (and equally easy to calculate) to let x be large too: $x = x_0 + vt$ for fixed v (maybe $v = 0$) and x_0. Since x and t only appear in the "outer" iterated integral it is enough to write

$$\psi(x_0 + vt, t) = \frac{2}{\epsilon} \int_{\mathbb{R}} \hat{\psi}_0(\lambda) e^{-2i\lambda x_0/\epsilon} e^{-2it(\lambda v + \lambda^2)/\epsilon} \, d\lambda.$$

This is an integral built for Kelvin's method of stationary phase. One simple stationary phase point $\lambda = \lambda_c := -v/2 = -(x - x_0)/(2t)$, so

$$\psi(x_0 + vt, t) = e^{-i\pi/4} \sqrt{\frac{2\pi}{\epsilon t}} \hat{\psi}_0(\lambda_c) e^{-2i\lambda_c x_0/\epsilon} e^{2it\lambda_c^2/\epsilon} + \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

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Another singular limit we may consider is $\epsilon \to 0$ (semiclassical limit). Now we must use iterated/double integral, but because Schrödinger's equation has an exponential Green's function it is useful to carefully exchange the order of integration and reduce the problem again to a single integral: for $t > 0$,

$$\psi(x,t) = \frac{e^{-i\pi/4}}{\sqrt{2\pi\epsilon t}} \int_{\mathbb{R}} e^{iI(y;x,t)/\epsilon} \sqrt{\rho_0(y)}\, dy, \quad I(y;x,t) := S(y) + \frac{(y-x)^2}{2t}$$

Again the method of stationary phase applies, now to the limit $\epsilon \downarrow 0$:

$$\psi(x,t) = \frac{1}{\sqrt{t}} \sum_{n=0}^{2P} \frac{e^{i\pi((-1)^n-1)/4}}{\sqrt{|I''(y_n;x,t)|}} \sqrt{\rho_0(y_n)} e^{iI(y_n;x,t)/\epsilon} + \mathcal{O}(\epsilon)$$

where $y_n = y_n(x,t)$, and $y_0 < y_1 < \cdots < y_{2P}$ are the stationary phase points, that is, the roots (assumed simple) of $I'(y;x,t) = 0$.

Dispersive Linear Waves

Prototype: free Schrödinger equation.

The condition that $y = y(x,t)$ is a stationary phase point is

$$I'(y;x,t) = u_0(y) + \frac{y-x}{t} = 0 \quad \Leftrightarrow \quad x = u_0(y)t + y.$$

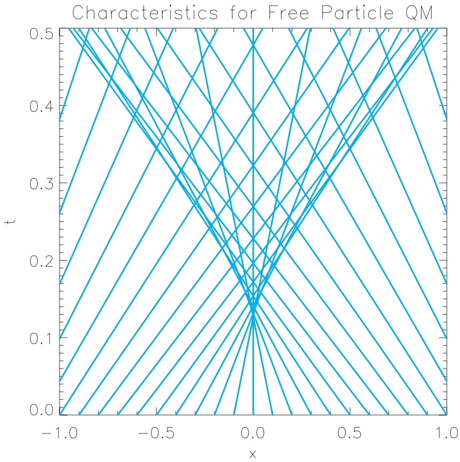
This is also the equation for intercepts y of characteristics through (x,t) for the formal limit of the Madelung system ($\rho := |\psi|^2$ and $u := \epsilon \mathrm{Im}(\psi_x/\psi)$)

$$\rho_t + (\rho u)_x = 0, \quad u_t + uu_x = \frac{\epsilon^2}{2}\left[\frac{\rho_{xx}}{2\rho} - \left(\frac{\rho_x}{2\rho}\right)^2\right]_x.$$

Dispersive Linear Waves

Prototype: free Schrödinger equation.

Here are the characteristic lines in the case $u_0(x) = -8\,\mathrm{sech}^2(x)\tanh(x)$:



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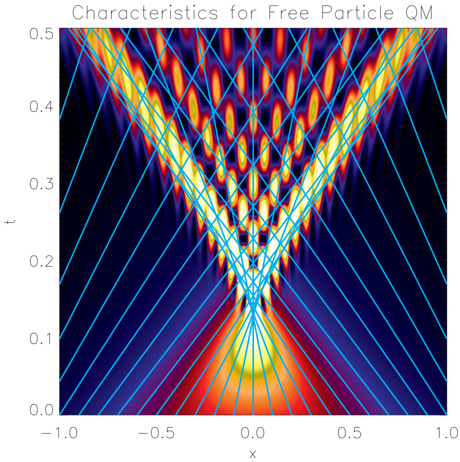


Dispersive Nonlinear Waves

Integrability: Fourier transform $\to$ inverse-scattering transform.

Similar precision of analysis is available in principle for nonlinear dispersive wave problems that are integrable:

- In place of the Fourier transform of the initial data, we have instead the *direct scattering transform*. Usually requires the analysis of a linear ODE (or PDE) with a *spectral parameter* to obtain scattering data (one or more functions of the spectral parameter).
- Just as in the linear theory, one has explicit exponential evolution of the scattering data in time t.
- In place of the inverse-Fourier transform of the time-evolved transform data, one has the *inverse-scattering transform*. Usually requires the solution of a linear Riemann-Hilbert problem (or $\overline{\partial}$ problem).

Let's illustrate these steps in a bit more detail for the defocusing nonlinear Schrödinger equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - |\psi|^2\psi = 0,\ \psi(x,0) = \sqrt{\rho_0(x)}e^{iS(x)/\epsilon},\ S(x) := \int_0^x u_0(y)\,dy.$$

The PDE is the compatibility condition for the two linear problems ($\lambda \in \mathbb{C}$ is the spectral parameter):

$$\epsilon\frac{\partial \mathbf{w}}{\partial x} = \mathbf{U}\mathbf{w}, \quad \mathbf{U} = \mathbf{U}(x,t,\lambda) := \begin{bmatrix} -i\lambda & \psi \\ \psi^* & i\lambda \end{bmatrix}$$

$$\epsilon\frac{\partial \mathbf{w}}{\partial t} = \mathbf{V}\mathbf{w}, \quad \mathbf{V} = \mathbf{V}(x,t,\lambda) := \begin{bmatrix} -i\lambda^2 - i\frac{1}{2}|\psi|^2 & \lambda\psi + i\frac{1}{2}\epsilon\psi_x \\ \lambda\psi^* - i\frac{1}{2}\epsilon\psi_x^* & i\lambda^2 + i\frac{1}{2}|\psi|^2 \end{bmatrix}.$$

Introducing real variables (Madelung, 1926)

$$\rho := |\psi|^2 \text{ and } u := \mathrm{Im}\left\{\frac{\epsilon\psi_x}{\psi}\right\} \implies \rho(x,0)=\rho_0(x) \text{ and } u(x,0)=u_0(x),$$

one can check that the defocusing nonlinear Schrödinger equation for ψ implies the following closed system of equations on ρ and u:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0 \quad \text{and} \quad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}u^2+\rho\right)=\frac{1}{2}\epsilon^2\frac{\partial F[\rho]}{\partial x}$$

where $F[\rho]$ denotes the expression

$$F[\rho] := \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2}-\left(\frac{1}{2\rho}\frac{\partial\rho}{\partial x}\right)^2 .$$

Neglecting $\epsilon^2 F_x$ leads to a closed, ϵ-independent hyperbolic system governing expected limits, the *dispersionless defocusing NLS system*.

We need to calculate the *Jost solution* $\mathbf{w}$ of the Zakharov-Shabat equation

$$\epsilon \frac{d\mathbf{w}}{dx} = \begin{bmatrix} -i\lambda & \sqrt{\rho_0(x)}e^{iS(x)/\epsilon} \\ \sqrt{\rho_0(x)}e^{-iS(x)/\epsilon} & i\lambda \end{bmatrix} \mathbf{w},$$

that is, the solution for $\lambda \in \mathbb{R}$ that is determined (assuming sufficiently rapid decay of ρ_0 for large $|x|$) by the conditions

$$\mathbf{w}(x) = \begin{bmatrix} e^{-i\lambda x/\epsilon} \\ 0 \end{bmatrix} + R_0^\epsilon(\lambda) \begin{bmatrix} 0 \\ e^{i\lambda x/\epsilon} \end{bmatrix} + o(1), \quad x \to +\infty$$

and

$$\mathbf{w}(x) = T_0^\epsilon(\lambda) \begin{bmatrix} e^{-i\lambda x/\epsilon} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty,$$

for some coefficients $R_0^\epsilon(\lambda)$ (the *reflection coefficient*) and $T_0^\epsilon(\lambda)$ (the *transmission coefficient*).

For the inverse transform, solve (for each fixed x and t) the following Riemann-Hilbert problem: seek $\mathbf{M} : \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}(2, \mathbb{C})$ such that:

- **Analyticity**: $\mathbf{M}$ is analytic in each half-plane, and takes boundary values $\mathbf{M}_{\pm} : \mathbb{R} \to \mathrm{SL}(2, \mathbb{C})$ on the real line from $\mathbb{C}_{\pm}$.
- **Jump Condition**: The boundary values are related by

$$\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda) \begin{bmatrix} 1 - |R_0^{\epsilon}(\lambda)|^2 & -e^{-2i(\lambda x + \lambda^2 t)/\epsilon} R_0^{\epsilon}(\lambda)^* \\ e^{2i(\lambda x + \lambda^2 t)/\epsilon} R_0^{\epsilon}(\lambda) & 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}.$$

- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

The solution of the initial-value problem is given by

$$\psi(x, t) = 2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda).$$

Semiclassical Defocusing NLS

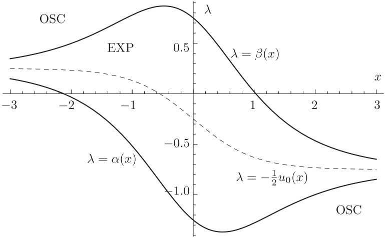
Semiclassical approximation of R_0^ϵ.

The first step is to calculate the reflection coefficient for small $\epsilon > 0$. This is a job for classical analysis, specifically the WKB method. Set:

$$\alpha(x) := -\frac{1}{2}u_0(x) - \sqrt{\rho_0(x)} \quad \text{and} \quad \beta(x) := -\frac{1}{2}u_0(x) + \sqrt{\rho_0(x)}.$$

Solutions of the Zakharov-Shabat problem are

- Rapidly oscillatory if $\lambda < \alpha(x)$ or $\lambda > \beta(x)$
- Exponentially growing or decaying if $\alpha(x) < \lambda < \beta(x)$.



Assume that there are at most two turning points $x_-(\lambda) \leq x_+(\lambda)$.

WKB analysis, plus connection analysis based on Airy functions near turning points, yields the following results:

- If $\lambda < \lambda_- := \inf_{x\in\mathbb{R}} \alpha(x)$ or $\lambda > \lambda_+ := \sup_{x\in\mathbb{R}} \beta(x)$, then $R_0^\epsilon(\lambda) = \mathcal{O}(\epsilon)$.
- If $\lambda \in (\lambda_-, \lambda_+)$, then:

$$R_0^\epsilon(\lambda) = e^{-2i\Phi(\lambda)/\epsilon}(1+\mathcal{O}(\epsilon)) \quad \text{and} \quad |T_0^\epsilon(\lambda)|^2 = e^{-2\tau(\lambda)/\epsilon}(1+\mathcal{O}(\epsilon))$$

where with $\sigma := \operatorname{sgn}(\lambda + \frac{1}{2}u_0(+\infty))$,

$$\begin{aligned}
\tau(\lambda) &:= \int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{\rho_0(y) - (\lambda + \tfrac{1}{2}u_0(y))^2}\, dy \\
\Phi(\lambda) &:= \tfrac{1}{2}S(x_+(\lambda)) + \lambda x_+(\lambda) \\
&\quad - \int_{x_+(\lambda)}^{+\infty} \left[\sigma\sqrt{(\lambda + \tfrac{1}{2}u_0(y))^2 - \rho_0(y)} - (\lambda + \tfrac{1}{2}u_0(y))\right] dy.
\end{aligned}$$

Since $|R_0^\epsilon(\lambda)|^2 + |T_0^\epsilon(\lambda)|^2 = 1$ holds, we will approximate $R_0^\epsilon(\lambda)$ by:

$$\tilde{R}_0^\epsilon(\lambda) := \chi_{[\lambda_-,\lambda_+]}(\lambda)\sqrt{1 - e^{-2\tau(\lambda)/\epsilon}}e^{-2i\Phi(\lambda)/\epsilon}, \quad \lambda \in \mathbb{R}.$$

We will show directly that after replacing $R_0^\epsilon(\lambda)$ with $\tilde{R}_0^\epsilon(\lambda)$:

- The Riemann-Hilbert problem can indeed be solved as long as ϵ is sufficiently small.
- When $t = 0$, the extracted potential $\tilde{\psi}$ is close in the limit $\epsilon \to 0$ to the actual initial data.

It can be shown (dressing method) that $\tilde{\psi}$ is also an exact solution of the defocusing nonlinear Schrödinger equation.

Aside: How to Solve a Riemann-Hilbert Problem

Associated singular integral equations

Let Σ be an oriented contour (perhaps with self-intersection points), and let $\mathbf{V} : \Sigma \to \mathrm{SL}(2, \mathbb{C})$ be a given jump matrix decaying to $\mathbb{I}$ as $\lambda \to \infty$ along any unbounded arcs of Σ. A general Riemann-Hilbert problem is the following: find $\mathbf{M} : \mathbb{C} \setminus \Sigma \to \mathrm{SL}(2, \mathbb{C})$ such that:

- **Analyticity**: $\mathbf{M}$ is analytic in its domain of definition, and takes boundary values $\mathbf{M}_\pm : \Sigma \to \mathrm{SL}(2, \mathbb{C})$ on Σ from the left $(+)$ and right $(-)$.
- **Jump Condition**: The boundary values are related by $\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda)\mathbf{V}(\lambda)$ for $\lambda \in \Sigma$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

This problem can be studied by converting it into a linear system of singular integral equations.

Subtract $\mathbf{M}_-(\lambda)$ from both sides of the jump condition:

$$\mathbf{M}_+(\lambda) - \mathbf{M}_-(\lambda) = \mathbf{M}_-(\lambda)(\mathbf{V}(\lambda) - \mathbb{I}), \quad \lambda \in \Sigma.$$

Taking into account the analyticity of $\mathbf{M}$ in $\mathbb{C} \setminus \Sigma$ and the asymptotic value of $\mathbb{I}$ as $\lambda \to \infty$ it is necessary that $\mathbf{M}(\lambda)$ is given by the Cauchy integral (Plemelj formula):

$$\mathbf{M}(\lambda) = \mathbb{I} + \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{M}_-(\mu)(\mathbf{V}(\mu) - \mathbb{I})}{\mu - \lambda} \, d\mu, \quad \lambda \in \mathbb{C} \setminus \Sigma.$$

Letting λ tend to Σ from the right we obtain a closed equation for the boundary value $\mathbf{M}_-(\lambda)$, $\lambda \in \Sigma$:

$$\mathbf{X}(\lambda) - \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{X}(\mu)(\mathbf{V}(\mu) - \mathbb{I})}{\mu - \lambda_-} \, d\mu = \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{V}(\mu) - \mathbb{I}}{\mu - \lambda_-} \, d\mu, \quad \lambda \in \Sigma,$$

where $\mathbf{X}(\lambda) := \mathbf{M}_-(\lambda) - \mathbb{I}$.

Aside: How to Solve a Riemann-Hilbert Problem

Associated singular integral equations

If the jump matrix $\mathbf{V}$ depends on parameters (e.g., x, t, ϵ), one can consider the asymptotic behavior of the Riemann-Hilbert problem with respect to one or more parameters. While one could attempt to analyze the singular equation, this would generally be a difficult (perhaps impossible) task, and we will proceed differently.

The singular integral equation is perhaps the most useful in the *small norm setting*. This means that $\mathbf{V} - \mathbb{I}$ is small in both the $L^2(\Sigma)$ and $L^\infty(\Sigma)$ sense. The utility of such estimates is a consequence of the fact that for a general class of contours Σ, the operator

$$\mathbf{F} \mapsto \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{F}(\mu)\, d\mu}{\mu - \lambda_-}$$

is bounded on $L^2(\Sigma)$, with a norm that only depends on geometrical details of Σ. See Coifman, McIntosh, Meyer for Lipschitz arcs, and Beals and Coifman for self-intersection points.

For problems of small norm type, the following hold true:

- The singular integral equation can be solved in $L^2(\Sigma)$ by iteration (contraction mapping, or Neumann series). This guarantees existence and uniqueness of the solution.
- It also allows the solution to be constructed (approximated with arbitrary accuracy and estimated). The $L^2(\Sigma)$ norm of $\mathbf{X}$ is proportional to that of $\mathbf{V} - \mathbb{I}$.
- Under suitable other technical assumptions, $\mathbf{M}(\lambda)$ has an asymptotic expansion as $\lambda \to \infty$:

$$\mathbf{M}(\lambda) = \mathbb{I} + \sum_{n=1}^{N} \lambda^{-n}\mathbf{M}_n + \mathcal{O}(\lambda^{-(N+1)}), \quad \lambda \to \infty$$

and the moments $\mathbf{M}_n$ are bounded in terms of norms of $\mathbf{V} - \mathbb{I}$.

Semiclassical Defocusing NLS

Riemann-Hilbert problem of inverse scattering.

Seek $\mathbf{M} : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathrm{SL}(2, \mathbb{C})$ with the following properties:

- **Analyticity**: $\mathbf{M}$ is analytic in its domain of definition and takes boundary values $\mathbf{M}_\pm(\lambda)$ on (λ_-, λ_+) from $\mathbb{C}_\pm$.
- **Jump Condition**: $\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda)\mathbf{V}(\lambda)$ for $\lambda_- < \lambda < \lambda_+$, where

$$\mathbf{V}(\lambda) = \begin{bmatrix} e^{-2\tau(\lambda)/\epsilon} & -e^{-2i\theta(\lambda;x,t)/\epsilon} H^\epsilon(\lambda) \\ e^{2i\theta(\lambda;x,t)/\epsilon} H^\epsilon(\lambda) & 1 \end{bmatrix},$$

  $\theta(\lambda; x, t) := \lambda x + \lambda^2 t - \Phi(\lambda)$, and $H^\epsilon(\lambda) := \sqrt{1 - e^{-2\tau(\lambda)/\epsilon}}$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

This is *not* a small-norm problem in the semiclassical limit $\epsilon \to 0$.

Let $g : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathbb{C}$ be analytic with $g(\infty) = 0$, and make the substitution $\mathbf{M}(\lambda) = \mathbf{N}(\lambda) e^{ig(\lambda)\sigma_3/\epsilon}$. Then $\mathbf{N} : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to SL(2, \mathbb{C})$ satisfies the conditions of this related Riemann-Hilbert problem:

- **Analyticity**: $\mathbf{N}$ is analytic in $\mathbb{C} \setminus [\lambda_-, \lambda_+]$, taking boundary values $\mathbf{N}_\pm(\lambda)$ on $[\lambda_-, \lambda_+]$ from $\mathbb{C}_\pm$.
- **Jump Condition**: The boundary values are related by $\mathbf{N}_+(\lambda) = \mathbf{N}_-(\lambda)\mathbf{V}^{(\mathbf{N})}(\lambda)$ for $\lambda_- < \lambda < \lambda_+$, where

$$\mathbf{V}^{(\mathbf{N})}(\lambda) := \begin{bmatrix} e^{2(\Delta(\lambda)-\tau(\lambda))/\epsilon} & -e^{-2i\phi(\lambda)/\epsilon} H^\epsilon(\lambda) \\ e^{2i\phi(\lambda)/\epsilon} H^\epsilon(\lambda) & e^{-2\Delta(\lambda)/\epsilon} \end{bmatrix},$$

  $2\Delta(\lambda) := -i(g_+(\lambda) - g_-(\lambda))$, and $2\phi(\lambda) := 2\theta(\lambda) - g_+(\lambda) - g_-(\lambda)$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{N}(\lambda) \to \mathbb{I}$.

We further suppose that $g(\lambda) = g(\lambda^*)^*$, making ϕ and Δ real.

Try to pick $g(\lambda)$ so that (λ_-, λ_+) splits into three types of subintervals:

- **Voids**: These are characterized by the conditions $\Delta(\lambda) \equiv 0$ and $\phi'(\lambda) > 0$.
- **Bands**: These are characterized by the conditions $0 < \Delta(\lambda) < \tau(\lambda)$ and $\phi'(\lambda) \equiv 0$.
- **Saturated regions**: These are characterized by the conditions $\Delta(\lambda) \equiv \tau(\lambda)$ and $\phi'(\lambda) < 0$.

We now examine the consequences of each type of interval for the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$.

Under the condition that $\Delta(\lambda) \equiv 0$, the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$ has an "upper-lower" factorization:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon} H^{\epsilon}(\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon} H^{\epsilon}(\lambda) & 1 \end{bmatrix}.$$

Let us assume that $\phi(\lambda)$ and $\tau(\lambda)$ are analytic (this will be the case if the initial data functions u_0 and ρ_0 are analytic). Then the condition $\phi'(\lambda) > 0$ makes $\phi(\lambda)$ a real analytic function that is strictly increasing in the void interval. By the Cauchy-Riemann equations, it follows that the imaginary part of $\phi(\lambda)$ is positive (negative) in the upper (lower) half-plane.

This implies that the first (second) matrix factor has an analytic continuation into the lower (upper) half-plane that is exponentially close to the identity matrix in the limit $\epsilon \to 0$.

The strict inequalities $0 < \Delta(\lambda) < \tau(\lambda)$ imply that the diagonal elements of $\mathbf{V}^{(\mathbf{N})}(\lambda)$, namely

$$e^{2(\Delta(\lambda)-\tau(\lambda))/\epsilon} \quad \text{and} \quad e^{-2\Delta(\lambda)/\epsilon}$$

are both exponentially small in the semiclassical limit $\epsilon \to 0$. The condition $\phi'(\lambda) \equiv 0$ together with the inequality $\tau(\lambda) > 0$ that holds for all $\lambda \in (\lambda_-, \lambda_+)$ then implies that $\mathbf{V}^{(\mathbf{N})}(\lambda)$ is exponentially close in the semiclassical limit to a constant off-diagonal matrix:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 0 & -e^{-2i\phi/\epsilon} \\ e^{2i\phi/\epsilon} & 0 \end{bmatrix} + \text{exponentially small terms.}$$

The real constant ϕ can be different for different bands, and it generally can depend on x and t (but not ϵ).

Deift-Zhou Steepest Descent Method

Saturated regions.

Under the condition that $\Delta(\lambda) \equiv \tau(\lambda)$, the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$ has a "lower-upper" factorization:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon} H^{\epsilon}(\lambda) & 1 \end{bmatrix} \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon} H^{\epsilon}(\lambda) \\ 0 & 1 \end{bmatrix}.$$

The condition $\phi'(\lambda) < 0$ makes $\phi(\lambda)$ a real analytic function that is strictly decreasing in the void interval. By the Cauchy-Riemann equations, it follows that the imaginary part of $\phi(\lambda)$ is negative (positive) in the upper (lower) half-plane.

This again implies that the first (second) matrix factor has an analytic continuation into the lower (upper) half-plane that is exponentially close to the identity matrix in the limit $\epsilon \to 0$.

Build g by (temporarily) ignoring the inequalities. Suppose that there are $N+1$ bands in (λ_-, λ_+) that we will denote by (a_j, b_j) with $\lambda_- < a_0 < b_0 < a_1 < b_1 < \cdots < a_N < b_N < \lambda_+$. The complementary intervals are either voids or saturated regions.

Recall that the boundary values of g are subject to the following:

- $g_+(\lambda) - g_-(\lambda) = 0$ which implies $g'_+(\lambda) - g'_-(\lambda) = 0$ for λ in voids and outside of $[\lambda_-, \lambda_+]$.
- $g'_+(\lambda) + g'_-(\lambda) = 2\theta'(\lambda)$ for λ in bands.
- $g_+(\lambda) - g_-(\lambda) = 2i\tau(\lambda)$ which implies $g'_+(\lambda) - g'_-(\lambda) = 2i\tau'(\lambda)$ for λ in saturated regions.

We therefore know $g'_+ - g'_-$ everywhere along $\mathbb{R}$ with the exception of the band intervals, where we know instead $g'_+ + g'_-$.

Fact: these conditions make up an easy (scalar) Riemann-Hilbert problem that can be solved explicitly for $g'(\lambda)$; the solution also puts $2N+2$ conditions on the $2N+2$ band endpoints.

If the equations on the unknowns $a_0, b_0, \ldots, a_N, b_N$ have a unique solution, then associated with the symbol sequence $(s_0, s_1, \ldots, s_{N+1})$, $s_n = \text{V}$ or $s_n = \text{S}$, indicating the types of complementary intervals in left-to-right order, $g(\lambda)$ is uniquely determined by integration.

Now recall the inequalities that the boundary values of g are supposed to satisfy. These inequalities should select:

- The value of N.
- The symbol sequence $(s_0, \ldots, s_{N+1})$.

The procedure in practice is therefore to determine N and $(s_0, \ldots, s_{N+1})$ so that the inequalities are true. The independent variables x and t are parameters in this procedure. In particular, $N = N(x, t)$.

Deift-Zhou Steepest Descent Method

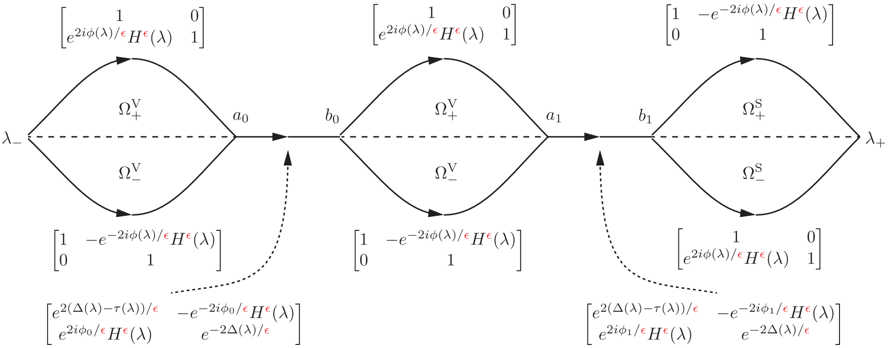
Steepest descent: opening lenses.

To exploit the matrix factorizations, let $\Omega_\pm^{\mathrm{V}}$ ($\Omega_\pm^{\mathrm{S}}$) denote the union of thin lens-shaped domains in $\mathbb{C}_\pm$ that abut voids (saturated regions). Define the piecewise analytic matrix function $\mathbf{L}$ by

$$\mathbf{L}(\lambda) := \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) & 1 \end{bmatrix}, & \lambda \in \Omega_+^{\mathrm{V}}, \\ \begin{bmatrix} 1 & 0 \\ -e^{2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) & 1 \end{bmatrix}, & \lambda \in \Omega_-^{\mathrm{S}}, \\ \begin{bmatrix} 1 & e^{-2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) \\ 0 & 1 \end{bmatrix}, & \lambda \in \Omega_-^{\mathrm{V}} \\ \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) \\ 0 & 1 \end{bmatrix}, & \lambda \in \Omega_+^{\mathrm{S}} \\ \mathbb{I}, & \text{otherwise.} \end{cases}$$

Make the substitution $\mathbf{N}(\lambda) = \mathbf{O}(\lambda)\mathbf{L}(\lambda)$. Then $\mathbf{O}(\lambda)$ satisfies the following Riemann-Hilbert problem...

- **Analyticity**: $\mathbf{O}$ is analytic in $\mathbb{C} \setminus \Sigma^{(\mathbf{O})}$, taking boundary values $\mathbf{O}_+$ ($\mathbf{O}_-$) on each oriented arc of $\Sigma^{(\mathbf{O})}$ from the left (right).
- **Jump Condition**: The boundary values are related by $\mathbf{O}_+(\lambda) = \mathbf{O}_-(\lambda)\mathbf{V}^{(\mathbf{O})}$ for $\lambda \in \Sigma^{(\mathbf{O})}$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{O}(\lambda) \to \mathbb{I}$.



Letting $\epsilon \to 0$ pointwise in λ along $\Sigma^{(\mathbf{O})}$, the jump matrix $\mathbf{V}^{(\mathbf{O})}(\lambda)$ converges to $\mathbb{I}$, except along each band (a_n, b_n), where

$$\mathbf{V}^{(\mathbf{O})}(\lambda) = \begin{bmatrix} 0 & -e^{-2i\phi_n/\epsilon} \\ e^{2i\phi_n/\epsilon} & 0 \end{bmatrix} + \text{exponentially small terms}$$

where ϕ_n are well-defined real-valued functions of (x, t) that are independent of λ and ϵ. For a formal approximation of $\mathbf{O}(\lambda)$, solve the following: seek $\dot{\mathbf{O}}^{(\mathrm{out})} : \mathbb{C} \setminus \text{bands} \to SL(2, \mathbb{C})$ with the properties

- **Analyticity**: $\dot{\mathbf{O}}^{(\mathrm{out})}$ is analytic where defined and takes boundary values $\dot{\mathbf{O}}^{(\mathrm{out})}_{\pm}(\lambda)$ from $\mathbb{C}_{\pm}$ on each band (a_n, b_n).
- **Jump Condition**: The boundary values satisfy $(n = 0, \ldots, N)$

$$\dot{\mathbf{O}}^{(\mathrm{out})}_{+}(\lambda) = \dot{\mathbf{O}}^{(\mathrm{out})}_{-}(\lambda) \begin{bmatrix} 0 & -e^{-2i\phi_n/\epsilon} \\ e^{2i\phi_n/\epsilon} & 0 \end{bmatrix}, \quad a_n < \lambda < b_n.$$

- **Normalization**: As $\lambda \to \infty$, $\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) \to \mathbb{I}$.

Deift-Zhou Steepest Descent Method

Parametrix construction.

Since the jump matrix is discontinuous at the band endpoints, we need to specify a singularity at each; we will suppose that for all n,

$$\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) = O((\lambda - a_n)^{-1/4}(\lambda - b_n)^{-1/4}), \quad \lambda \to a_n, b_n.$$

With this condition, there is a unique solution for $\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)$ that we call the *outer parametrix*. In general, it is constructed in terms of Riemann theta functions of genus N, but for $N = 0$ (one band) the solution is elementary:

$$\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) = e^{-i\phi_0\sigma_3/\epsilon}\mathbf{A}\gamma(\lambda)^{\sigma_3}\mathbf{A}^{-1}e^{i\phi_0\sigma_3/\epsilon}, \quad \text{where} \quad \mathbf{A} := \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

and where $\gamma(\lambda)$ is the function analytic for $\lambda \in \mathbb{C} \setminus [a_0, b_0]$ that satisfies

$$\gamma(\lambda)^4 = \frac{\lambda - b_0}{\lambda - a_0} \quad \text{and} \quad \lim_{\lambda \to \infty} \gamma(\lambda) = 1.$$

The approximation of the jump matrix by piecewise constants is inaccurate near the band endpoints. In disks $D_{a_0}, \ldots, D_{b_N}$ centered at the endpoints, it is necessary to use different (local) approximations of $\mathbf{O}$ called *inner parametrices*. These are built out of Airy functions.

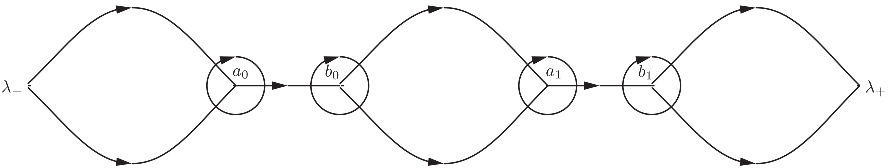
Combining inner and outer parametrices gives rise to an explicit, ad-hoc approximation of $\mathbf{O}(\lambda)$ called the *global parametrix* denoted $\dot{\mathbf{O}}(\lambda)$ and defined piecewise:

$$\dot{\mathbf{O}}(\lambda) := \begin{cases} \dot{\mathbf{O}}^{(\mathrm{in},D_p)}(\lambda), & \lambda \in D_p, \quad p = a_0, \ldots, b_N, \\ \dot{\mathbf{O}}^{(\mathrm{out})}(\lambda), & \text{otherwise.} \end{cases}$$

Let the *error* of the approximation be defined as the matrix function

$$\mathbf{E}(\lambda) := \mathbf{O}(\lambda)\dot{\mathbf{O}}(\lambda)^{-1}$$

wherever both factors make sense. This makes $\mathbf{E}(\lambda)$ analytic on the complement of an arcwise oriented contour $\Sigma^{(\mathbf{E})}$ (pictured).



While $\mathbf{O}$ is only specified as the solution of a Riemann-Hilbert problem, the global parametrix $\dot{\mathbf{O}}(\lambda)$ is known. Therefore we may regard the mapping $\mathbf{O} \to \mathbf{E}$ as a substitution resulting in an equivalent Riemann-Hilbert problem for $\mathbf{E}$.

Deift-Zhou Steepest Descent Method

Error analysis by small norm theory.

Since both $\mathbf{O}(\lambda) \to \mathbb{I}$ (by normalization condition) and $\dot{\mathbf{O}}(\lambda) \to \mathbb{I}$ (by construction) as $\lambda \to \infty$, we also must have $\mathbf{E}(\lambda) \to \mathbb{I}$ in this limit.

By direct calculations, one checks that as a consequence of the uniform boundedness of the outer parametrix outside all disks,

$$\mathbf{E}_+(\lambda) = \mathbf{E}_-(\lambda)(\mathbb{I} + o(1)) \quad \text{uniformly for } \lambda \in \Sigma^{(\mathbf{E})}.$$

This means that $\mathbf{E}(\lambda)$ satisfies the conditions of a Riemann-Hilbert problem of small norm type. Small-norm theory therefore implies that:

- $\mathbf{E}(\lambda)$ exists for sufficiently small ϵ and is unique, and hence (by unraveling the explicit substitutions) the same is true of $\mathbf{M}(\lambda)$.
- $\mathbf{E}(\lambda)$ has a Laurent series (convergent, because $\Sigma^{(\mathbf{E})}$ is bounded)

$$\mathbf{E}(\lambda) = \mathbb{I} + \sum_{n=1}^{\infty} \mathbf{E}_n \lambda^{-n} \quad \text{with} \quad \mathbf{E}_n = o(1), \quad \forall n.$$

To calculate $\tilde{\psi}(x,t)$, recall that $\mathbf{L}(\lambda)=\mathbb{I}$ and $\dot{\mathbf{O}}(\lambda)=\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)$ both hold for large enough $|\lambda|$, and that $g(\lambda)\to 0$ as $\lambda\to\infty$. Therefore,

$$
\begin{aligned}
\tilde{\psi}(x,t) &= 2i\lim_{\lambda\to\infty}\lambda M_{12}(\lambda)\\
&= 2i\lim_{\lambda\to\infty}\left[\mathbf{E}(\lambda)\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon}\right]_{12}\\
&= 2iE_{1,12}+2i\dot{O}^{(\mathrm{out})}_{1,12}\\
&= 2i\dot{O}^{(\mathrm{out})}_{1,12}+o(1).
\end{aligned}
$$

When $N=0$ (one band, (a_0,b_0)), this reads simply

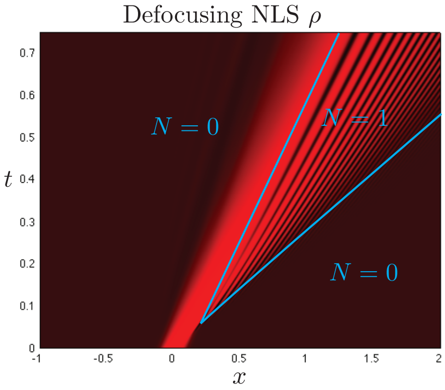
$$
\tilde{\psi}(x,t)=\frac{1}{2}(b_0-a_0)e^{-2i\phi_0/\epsilon}+o(1),\qquad \frac{\partial\phi_0}{\partial x}=\frac{1}{2}(a_0+b_0).
$$

This case can be proven to hold true for $|t|$ sufficiently small, and when $t=0$, $a_0=\alpha(x)$ and $b_0=\beta(x)$.

Semiclassical Defocusing NLS

Genus bifurcations for larger t.

For larger t, the g-function theory tiles the (x, t)-plane with regions corresponding to different genera N. The earliest point of transition is the shock time for the dispersionless NLS system.



$$\rho_0(x) = \frac{1}{10} + \frac{1}{2}e^{-256x^2}$$
$$u_0(x) = 1$$
$$\epsilon = 0.0122$$

Periodic boundary conditions

Genus bifurcations in the g-function are the integrable nonlinear analogues of stationary phase point bifurcations in the linear theory.

The Sine-Gordon Equation

Pure impulse Cauchy problem.

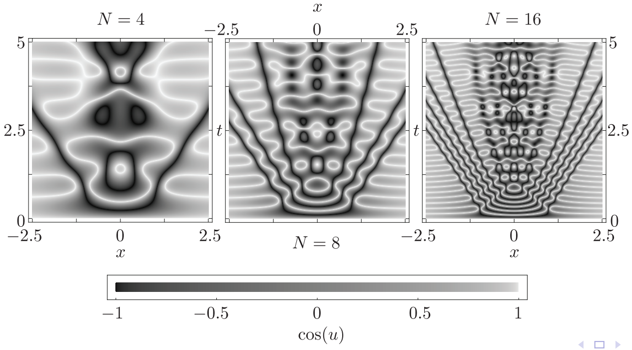
The sine-Gordon equation (in “laboratory coordinates”) is:

$$\epsilon^2 u_{tt} - \epsilon^2 u_{xx} + \sin(u) = 0.$$

A “pure impulse” Cauchy problem on $\mathbb{R}$ has initial data

$$u(x, 0; \epsilon) = 0, \quad \epsilon u_t(x, 0; \epsilon) = G(x), \quad x \in \mathbb{R},$$

where G is a given “impulse profile;” note that $|G(x)| = 2$ is a threshold for libration/rotation:



Here,

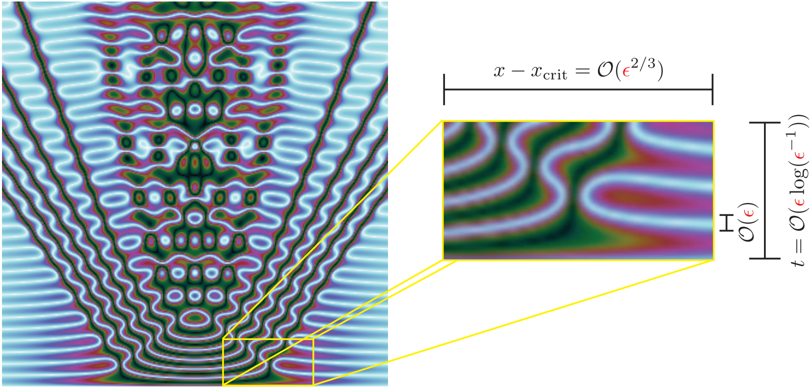
$$G(x) = -3\operatorname{sech}(x),$$

and

$$\epsilon = 3/(4N)$$

for $N = 4, 8, 16$.

The behavior of $u(x,t;\epsilon)$ is universal near critical points where $|G(x_c)| = 2$, provided $G'(x_c) \neq 0$:



Let $\nu := [12G'(x_c)]^{-1} > 0$ and set $\Delta x := x - x_c$.

Set $\mathcal{U}_0(y) := 1$ and $\mathcal{V}_0(y) := -y/6$. Generate $\{\mathcal{U}_m, \mathcal{V}_m\}_{m\in\mathbb{Z}}$ by the recursions

$$\mathcal{U}_{m+1}(y) := -\frac{1}{6}y\mathcal{U}_m(y) - \frac{\mathcal{U}_m'(y)^2}{\mathcal{U}_m(y)} + \frac{1}{2}\mathcal{U}_m''(y) \quad \text{and} \quad \mathcal{V}_{m+1}(y) := \frac{1}{\mathcal{U}_m(y)}$$

$$\mathcal{U}_{m-1}(y) := \frac{1}{\mathcal{V}_m(y)} \quad \text{and} \quad \mathcal{V}_{m-1}(y) := \frac{1}{2}\mathcal{V}_m''(y) - \frac{\mathcal{V}_m'(y)^2}{\mathcal{V}_m(y)} - \frac{1}{6}y\mathcal{V}_m(y).$$

It turns out that $(\mathcal{U}, \mathcal{V}) = (\mathcal{U}_m, \mathcal{V}_m)$ satisfy for each m the coupled system of second-order Painlevé II-type equations

$$\mathcal{U}''(y) + 2\mathcal{U}(y)^2\mathcal{V}(y) + \frac{1}{3}y\mathcal{U}(y) = 0$$
$$\mathcal{V}''(y) + 2\mathcal{U}(y)\mathcal{V}(y)^2 + \frac{1}{3}y\mathcal{V}(y) = 0.$$

Theorem (Buckingham & M., *J. Anal. Math.*, 118, 2012.)

Fix an integer m and assume that (x,t) lies in the horizontal strip S_m in the (x,t)-plane given by the inequality $|t - \frac{2}{3}m\epsilon\log(\epsilon^{-1})| \le \frac{1}{3}\epsilon\log(\epsilon^{-1})$. Suppose also that $\Delta x = \mathcal{O}(\epsilon^{2/3})$. Then

$$\cos(\tfrac{1}{2}\tilde{u}(x,t;\epsilon)) = (-1)^m \operatorname{sgn}(\mathcal{U}_m(y))\operatorname{sech}(T_{\mathsf{K}}) + E_{\cos}(x,t;\epsilon)$$
$$\sin(\tfrac{1}{2}\tilde{u}(x,t;\epsilon)) = (-1)^{m+1}\tanh(T_{\mathsf{K}}) + E_{\sin}(x,t;\epsilon)$$

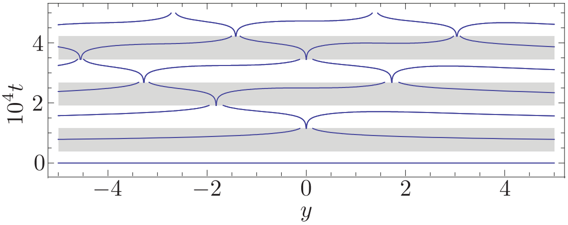
where $E_{\cos}$ and $E_{\sin}$ are small error terms vanishing[a] as $\epsilon \to 0$,

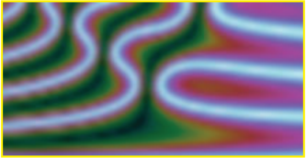
$$T_{\mathsf{K}} := \frac{t}{\epsilon} - 2m\log\left(\frac{4\nu^{1/3}}{\epsilon^{1/3}}\right) + \log|\mathcal{U}_m(y)|, \quad \textit{and} \quad y := \frac{\Delta x}{2\nu^{1/3}\epsilon^{2/3}}.$$

[a] except near distinguished points associated with singularities of $\log|\mathcal{U}_m|$.

The Sine-Gordon Equation
Double-scaling limit.

The universal wave pattern that appears can be described roughly as “kinks centered along real graphs of rational Painlevé-II functions”:





The basic roadmap of the proof is as follows:

- Spectral data associated with the impulse profile G is calculated using WKB methods. The essential data is fully discrete.
- After some analytic interpolation steps, the inverse-scattering problem can be written as a matrix Riemann-Hilbert problem, albeit one on a complicated contour with self-intersections.

The Sine-Gordon Equation

Double-scaling limit.

- Via systematic preparations (introduction of an appropriate g-function, Deift-Zhou steepest descent method) this problem is converted into another one with a jump matrix converging uniformly to something simple except near one point $w = w_*$. Away from this point there is an obvious approximate solution indexed by an arbitrary parameter $m \in \mathbb{Z}$.
- A local parametrix required near the exceptional point brings in the rational Painlevé-II functions with index m, and the kinks.

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Thank You!