Hamiltonian partial differential equations and Painlevé transcendents

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Lecture 2
Recall: the main goal is to compare the properties of solutions to the perturbed system

$$u_t = A(u)u_x + \epsilon A_2(u; u_x, u_{xx}) + \epsilon^2 A_3(u; u_x, u_{xx}, u_{xxx}) + \ldots$$

$$u(x, 0; \epsilon) = \varphi(x)$$

with solutions to the “dispersionless limit” $$\epsilon \to 0$$

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$$v(x, 0) = \varphi(x)$$
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\[ v(x, 0) = \varphi(x) \]

\[ H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \]
Of particular interest is the comparison near the point of gradient catastrophe of the dispersionless system

\[ v_t = A(v)v_x \]

e.g., such a point \((x_0, t_0)\) that the solution exists for \(t < t_0\), there exists the limit \(\lim_{t\to t_0} v(x, t)\)

but, for some \(x_0\)

\[ v_x(x, t), \quad v_t(x, t) \to \infty \quad \text{for} \quad (x, t) \to (x_0, t_0) \]

The problem: to describe the asymptotic behaviour of the generic solution \(u(x, t; \epsilon), \quad u(x, 0; \epsilon) = u_0(x)\)

to the perturbed system for \(\epsilon \to 0\)

in a neighborhood of the point of catastrophe \((x_0, t_0)\)
Phase transition in KdV equation

\[ u_t + uu_x + \epsilon^2 u_{xxx} = 0 \]
Main Conjecture (B.D., 2005): a finite list of types of the critical behaviour
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(Universality)
KdV asymptotics

\[ u_t + u u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \]

\[ u(x, t = 0, \epsilon) = \varphi(x) \]

\[ u(x, t, \epsilon) \approx U_{\text{ell}} \left( \frac{S(x, t)}{\epsilon}; r_1(x, t), r_2(x, t), r_3(x, t) \right) \]

\[ U_{\text{ell}}(z; r_1, r_2, r_3) \]

Weierstrass elliptic function

\[ (U'_{\text{ell}})^2 + 4(U_{\text{ell}} - r_1)(U_{\text{ell}} - r_2)(U_{\text{ell}} - r_3) = 0 \]

\[ u(x, t, \epsilon) = v(x, t) + O(\epsilon) \]

where

\[ v_t + v v_x = 0 \]

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KdV asymptotics

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\[ u(x, t, \epsilon) \simeq U_{\text{ell}} \left( \frac{S(x, t)}{\epsilon}; r_1(x, t), r_2(x, t), r_3(x, t) \right) \]

Where

\[ U_{\text{ell}}(z; r_1, r_2, r_3) \]

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A. Gurevich, L. Pitayevski '73

P. Lax, D. Levermore '83
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point of gradient catastrophe for the Hopf solution
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point of gradient catastrophe for the Hopf solution \( v(x, t) \)

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point of gradient catastrophe for the Hopf solution

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point of gradient catastrophe for the Hopf solution \( v(x, t) \)

small oscillations

Painlevé-II

\[ u(x, t, \epsilon) = v(x, t) + O(\epsilon) \]

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point of gradient catastrophe for the Hopf solution \( v(x, t) \)

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point of gradient catastrophe for the Hopf solution \( v(x, t) \)

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P.Lax, D.Levermore ‘83

T.Claeys, T.Grava

small oscillations

Painlevé-II

solitonic asymptotics

where

\[ v_t + v v_x = 0 \]

\[ v(x, 0) = \varphi(x) \]
Universality Conjecture, case of scalar PDEs

\[ u_t + a(u)u_x + \epsilon^2 \left[ c(u)u_{xxx} + b(u)u_{xx}u_x + d(u)u_x^3 \right] + O(\epsilon^3) = 0 \]

\[ u(x, 0; \epsilon) = \varphi(x) \]

Compare with solutions to the unperturbed equation with the same initial condition

\[ v_t + a(v) v_x = 0 \]

\[ v(x, 0) = \varphi(x) \]

Suppose \( v(x, t) \) has a generic gradient catastrophe at \( (x_0, t_0) \)

\[ v_0 := v(x_0, t_0) \]
**Part 1.** For $t < t_0$

$$u(x, t; \epsilon) \to v(x, t) \quad \text{as} \quad \epsilon \to 0$$

**Part 2.** There exists $\delta = \delta(\epsilon) > 0$ such that the solution $u(x, t; \epsilon)$ can be extended till $t = t_0 + \delta$

**Part 3.** The limit

$$\lim_{\epsilon \to 0} \alpha \epsilon^{-2/7} [u(x, t; \epsilon) - v_0]$$

exists and depends neither on the choice of generic solution nor on the choice of generic Hamiltonian perturbation.

$$x = x_0 + a_0 \gamma \epsilon^{4/7} T + \beta \epsilon^{6/7} X$$

$$t = t_0 + \gamma \epsilon^{4/7} T$$

where

$$\alpha = \kappa^{2/7} \rho^{-1/7}, \quad \beta = \kappa^{1/7} \rho^{3/7}, \quad \gamma = \kappa^{3/7} \rho^{2/7}$$

$$\kappa = -(a_0''' t_0 + f'''')$$

where $f(\varphi(x)) \equiv x$, $\rho = \frac{12c_0}{a_0'}$
Part 1. For $t < t_0$

$$u(x, t; \epsilon) \to v(x, t) \quad \text{as} \quad \epsilon \to 0$$

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$$\lim_{\epsilon \to 0} \alpha \epsilon^{-2/7} [u(x, t; \epsilon) - v_0] =: U(X, T)$$

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$$\kappa = -(a_0''' t_0 + f'''') \quad \text{where} \quad f(\varphi(x)) \equiv x, \quad \rho = \frac{12c_0}{a_0'’}$$
Equivalently, for the case of scalar Hamiltonian PDEs at the point of phase transition \( x = x_0, t = t_0, u = u_0 \), the generic solution has the following asymptotics:

\[
u(x, t) = v_0 + \frac{e^{2/7}}{\alpha} U \left( \frac{x - a_0(t - t_0) - x_0}{\beta e^{6/7}}, \frac{t - t_0}{\gamma e^{4/7}} \right) + O\left(e^{4/7}\right)
\]

where \( U(X, T) \) is a particular solution to the ODE

\[
P_I^2 X = T U - \left[ \frac{1}{6} U^3 + \frac{1}{24} (U'^2 + 2U U'') + \frac{1}{240} U^{IV} \right]
\]

(a differential equation in \( X \) depending on the parameter \( T \)).
Proof of Part I (small time behaviour):

D.Masoero and A.Raimondo

\[ u_t = a(u)u_x + \sum_{i=1}^{k} \epsilon_i u^{(2i+1)} \]

\[ u(x, 0; \epsilon) = \varphi(x) \in H^s, \quad s \geq 2k + 1 \]

Thm. The solution is continuous in \( \epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{R}^k \)
The smooth solution \( U(X, T) \) to \( P_I^2 \)

\[
X = TU - \left[ \frac{1}{6} U^3 + \frac{1}{24} (U'^2 + 2U U'') + \frac{1}{240} U^{IV} \right]
\]

Proof of existence: T. Claeys, M. Vanlessen, 2007
Comparison of KdV with $P_I^2$
A non-integrable case: the **Kawahara** equation

\[ u_t + 6u u_x + \epsilon^2 u_{xxx} - \epsilon^4 u_{xxxxx} = 0 \]

versus \( P_I^{2} \)
Motivations (for a scalar Hamiltonian equation)

\[ u_t + a(u) u_x + \epsilon^2 \left[ c(u) u_{xxx} + b(u) u_{xx} u_x + d(u) u_x^3 \right] + \mathcal{O}(\epsilon^3) = 0 \]

The solution to a Cauchy problem for the leading term

\[ v_t + a(v) v_x = 0 \]
\[ v(x, 0) = \varphi(x) \]

can be written in the implicit form

\[ x = a(v) t + f(v), \quad f(\varphi(x)) \equiv x \]

Point of gradient catastrophe \((x_0, t_0, v_0)\) such that

\[ x_0 = a(v_0) t_0 + f(v_0) \]
\[ 0 = a'(v_0) t_0 + f'(v_0) \]
\[ 0 = a''(v_0) t_0 + f''(v_0) \]
Local structure near the point of catastrophe

\[ v(x, t_0) \]

\[ (x_0, v_0) \]
Step 1: near the generic point of gradient catastrophe the solution to the dispersionless equation can be approximated by

\[ \bar{x} + a'_0 \bar{v} \bar{t} \approx \frac{1}{6} f'''(\bar{v})^3 \]

where

\[ \bar{x} = x - x_0 - a_0(t - t_0) \]
\[ \bar{t} = t - t_0 \]
\[ \bar{v} = v - v_0 \]

\[ a_0 = a(v_0), \quad a'_0 = a'(v_0) \quad \text{etc.} \]

After rescaling

\[ \bar{x} \mapsto \lambda \bar{x} \]
\[ \bar{t} \mapsto \lambda^{2/3} \bar{t} \]
\[ \bar{v} \mapsto \lambda^{1/3} \bar{v} \]

one obtains the above cubic equation modulo \[ \mathcal{O}\left(\lambda^{1/3}\right), \quad \lambda \to 0 \]
Step 2: replacing the original dispersive equation by KdV near the point \((x_0, t_0, v_0)\)

\[ u_t + a(u)u_x + \varepsilon^2 \left[ c(u)u_{xxx} + b(u)u_{xx}u_x + d(u)u_x^3 \right] + \mathcal{O}(\varepsilon^3) = 0 \]

Galilean transformation + rescaling

\[
\begin{align*}
\bar{x} &= x - x_0 - a_0 (t - t_0) &\mapsto \lambda \bar{x} \\
\bar{t} &= t - t_0 &\mapsto \lambda^{2/3} \bar{t} \\
\bar{u} &= u - v_0 &\mapsto \lambda^{1/3} \bar{u} \\
\varepsilon &\mapsto \lambda^{7/6} \varepsilon
\end{align*}
\]

yields

\[
\bar{u}_\bar{t} + a_0' \bar{u} \bar{u}_\bar{x} + \varepsilon^2 c_0 \bar{u}_{\bar{x}\bar{x}\bar{x}} \simeq 0, \quad c_0 = c(v_0)
\]

modulo \(\mathcal{O}\left(\lambda^{1/3}\right)\), \(\lambda \rightarrow 0\)
The calculation: after rescaling \( u_t \mapsto \lambda^{-1/3} u_t \)

A monomial in the rhs \( \epsilon^m u^{(i_1)} \ldots u^{(i_k)} \)

\[ i_1 + \cdots + i_k = m + 1 \]

So

\[ \epsilon^m u^{(i_1)} \ldots u^{(i_k)} \mapsto \lambda^{\frac{7}{6} m + \frac{1}{3} k - (i_1 + \cdots + i_k)} \epsilon^m u^{(i_1)} \ldots u^{(i_k)} \]

\[ = \lambda^{\frac{m}{6} + \frac{k}{3} - 1} \epsilon^m u^{(i_1)} \ldots u^{(i_k)} \]

Must have \( \frac{m}{6} + \frac{k}{3} - 1 \leq -\frac{1}{3} \) or \( m + 2k \leq 4 \)

Since \( m \geq 2 \) \( \Rightarrow m = 2, \ k = 1 \)

Only one monomial \( \epsilon^m u^{(i_1)} \ldots u^{(i_k)} = \epsilon^2 u_{xxx} \)
Step 3: choosing a particular solution to KdV

The trick: to replace PDE (the KdV) + initial condition by an ODE ("string equation")

\[ u_t + u u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \]
\[ u(x, 0; \epsilon) = \varphi(x) \]

First, rewrite the solution
\[ x = v t + f(v), \quad f(\varphi(x)) \equiv x \]
to Hopf equation
\[ v_t + v v_x = 0 \]
in the form
\[ \frac{\partial}{\partial v} \left[ F(v) + t \frac{v^2}{2} - x v \right] = 0, \quad F'(v) = f(v) \]
The idea: to determine the solution to the KdV equation

\[ u_t + u u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \]

\[ u(x, 0; \epsilon) = u_0(x) \]

with the same, modulo \( \mathcal{O}(\epsilon^2) \) initial data from the Euler-Lagrange equation

\[ \frac{\delta}{\delta u(x)} \left\{ H_F[u; \epsilon] + \int \left( t \frac{u^2}{2} - x u \right) dx \right\} = 0 \]

(“string equation”)
Lemma. Given a first integral $H_F[u]$ of the KdV equation then the space of solutions to the string equation

$$x = t u + \frac{\delta H_F[u]}{\delta u(x)}$$

is invariant wrt the KdV flow

Recall: Lax-Novikov lemma (1974):

Lax version: given a first integral $H[u]$ of KdV then the set of stationary points

$$\frac{\delta H[u]}{\delta u(x)} = 0$$

is invariant wrt the KdV flow
Novikov version: given a first integral $H[u]$ of the KdV then the space of stationary points $u_s = 0$
the space of stationary points
of the Hamiltonian flow

$$u_s = \frac{\partial}{\partial x} \frac{\delta H[u]}{\delta u(x)}$$
is invariant wrt the KdV flow

$$u_t = \frac{\partial}{\partial x} \frac{\delta H_{\text{KdV}}[u]}{\delta u(x)}$$

Indeed, from commutativity of Hamiltonians

$$\{H_{\text{KdV}}, H\} = 0$$
it follows commutativity of the flows $(u_t)_s = (u_s)_t$

So, if the KdV initial data satisfies $u_s = 0$, then so does the solution. Observe

$$u_s = 0 \iff \frac{\delta H[u]}{\delta u(x)} = \text{const} \iff \frac{\delta \tilde{H}[u]}{\delta u(x)} = 0$$

where $\tilde{H}[u] = H[u] - \text{const} \int u \, dx$
Galilean symmetry for KdV equation

\[
\begin{align*}
x & \mapsto x - c t \\
t & \mapsto t \\
u & \mapsto u + c
\end{align*}
\]

Infinitesimal form

\[
u_\tau = 1 - t u_x, \quad (u_\tau)_t = (u_t)_\tau
\]

So, the string equation

\[
x = t u + \frac{\delta H_F[u]}{\delta u(x)}
\]

describes the stationary points of the linear combination

\[
\left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial s} \right) u = 0
\]

Here

\[
u_s = \frac{\partial}{\partial x} \frac{\delta H_F[u]}{\delta u(x)}
\]
Construction of the functional $H_{F}[u; \epsilon]$ uses the theory of deformations of the conservation laws (see the previous lecture).

For the Hopf equation $v_t + v v_x = 0$ the functional

$$H_{F}^{0} = \int F(v) \, dx$$

is a conservation law for an arbitrary function $F$. 
Theorem. For any function $F$ there exists a deformed functional

$$H_F = \int \left[ F(u) - \frac{\epsilon^2}{24} F'''(u) u_x^2 + \epsilon^4 \left( \frac{1}{480} F^{(4)} u_{xx}^2 - \frac{1}{3456} F^{(6)} u_x^4 \right) + \ldots \right] \, dx$$

being a conservation law for the KdV equation

$$u_t + uu_x + \frac{\epsilon^2}{12} u_{xxx} = 0$$
An explicit formula in terms of Lax operator

\[ L = \frac{\epsilon^2}{2} \frac{d^2}{dx^2} + u(x) \]

Then

\[ H_F = \int h_F \, dx \]

where

\[ h_F = \text{res} \, F^{(1/2)}(L) \]
So, one arrives at studying solutions to the Euler-Lagrange equation

\[
\frac{\delta}{\delta u(x)} \left\{ H_F[u; \epsilon] + \int \left( t \frac{u^2}{2} - x u \right) dx \right\} = 0
\]

where

\[
H_F = \int \left[ F(u) - \frac{\epsilon^2}{24} F'''(u)u_x^2 + \epsilon^4 \left( \frac{1}{480} F^{(4)}u_{xx}^2 - \frac{1}{3456} F^{(6)}u_x^4 \right) + \ldots \right] dx
\]

Explicitly

\[
x = u t + f(u) + \frac{\epsilon^2}{24} \left[ 2f''(u)u_{xx} + f'''(u)u_x^2 \right] + \frac{\epsilon^4}{240} f'''(u)u_{xxx} + \ldots
\]
The last step: to apply to the “string equation”

\[ x = ut + f(u) + \frac{\epsilon^2}{24} \left[ 2f''(u)u_{xx} + f'''(u)u_x^2 \right] + \frac{\epsilon^4}{240} f'''(u)u_{xxxx} + \ldots \]

a rescaling near the point of phase transition

\[
\begin{align*}
\bar{x} &= x - x_0 - v_0(t - t_0) & \mapsto & \lambda \bar{x} \\
\bar{t} &= t - t_0 & \mapsto & \lambda^{2/3} \bar{t} \\
\bar{\bar{u}} &= u - v_0 & \mapsto & \lambda^{1/3} \bar{\bar{u}} \\
\epsilon & \mapsto & \lambda^{7/6} \epsilon
\end{align*}
\]

to arrive at

\[
\bar{x} = \bar{\bar{u}} \bar{t} + \frac{1}{6} f_0''' \left[ \bar{\bar{u}}^3 + \frac{\epsilon^2}{4} \left( 2\bar{\bar{u}} \bar{\bar{u}}_{\bar{x}\bar{x}} + \bar{\bar{u}}_{\bar{x}}^2 \right) + \frac{\epsilon^4}{40} \bar{\bar{u}}_{\bar{x}\bar{x}\bar{x}\bar{x}} \right] + O \left( \lambda^{1/3} \right)
\]

Choosing \( \lambda = \epsilon^{6/7} \) one obtains \( P_I^2 \)
Solutions to $P^2_I$

$$X = T U - \left[ \frac{1}{6} U^3 + \frac{1}{24} (U'^2 + 2UU'') + \frac{1}{240} U^{IV} \right]$$

satisfy KdV

$$U_T + U U_X + \frac{1}{12} U_{XXX} = 0$$

Matching condition: for large $|X|$

$$U(X, T) \sim \text{(unique) root of cubic equation} \quad X = U T - \frac{1}{6} U^3$$

Matching + smoothness $\Rightarrow$ choice of a particular solution

Uses:
• Riemann-Hilbert formulation of inverse scattering
• asymptotics of the scattering data of Lax operator

\[ L = \frac{\epsilon^2}{2} \frac{d^2}{dx^2} + u \]

• Deift-Zhou asymptotic analysis of the Riemann-Hilbert problem
Second order systems

\[ u_t = A(u)u_x + \epsilon A_2(u; u_x, u_{xx}) + \epsilon^2 A_3(u; u_x, u_{xx}, u_{xxx}) + \ldots \]

1) Hyperbolic case: eigenvalues of \( A(u) \) are real and distinct. Riemann invariants diagonalize the leading term

\[
\begin{align*}
    r_t^+ + \lambda^+(r)r_x^+ + O(\epsilon) &= 0 \\
    r_t^- + \lambda^-(r)r_x^- + O(\epsilon) &= 0
\end{align*}
\]

For the leading order system only one of Riemann invariants breaks down. Critical behaviour described by the same solution to \( P^2_I \)

2) Elliptic case. The Riemann invariants are complex conjugate. They break down simultaneously
Dispersionless case, characteristic directions $x_+, x_-$

$$\frac{\partial r_+}{\partial x_-} = 0, \quad \frac{\partial r_-}{\partial x_+} = 0$$

At the critical point: Whitney fold catastrophe

$$\begin{cases} 
  x_+ = r_+ \\
  x_- = r_+ r_- - \frac{1}{6} r_-^3
\end{cases}$$

(hyperbolic case)

In the elliptic case $x_- = \bar{x}_+$, hence the Cauchy-Riemann eqs.

$$\frac{\partial r_+}{\partial \bar{x}_+} = 0$$

At the critical point $x_+ = r_+^2$

(elliptic umbilic singularity)
How to solve dispersionless equations?

Example of (focusing) NLS

\[
H_0 = \frac{1}{2} \int (u^2 - u v^2) \, dx
\]

First integrals

\[
F_0 = \int f(u, v) \, dx
\]

\[
0 = \{H_0, F_0\} = \int \left[ \frac{\partial h_0}{\partial u} \frac{\partial f_0}{\partial v} + \frac{\partial h_0}{\partial v} \frac{\partial f_0}{\partial u} \right] \, dx \quad \leftrightarrow \quad f_{vv} + u f_{uu} = 0
\]
Lemma. Given a first integral $f$, the system

\[
\begin{align*}
x &= f_u(u, v) \\
t &= f_v(u, v)
\end{align*}
\]

defines a solution to dispersionless NLS. Any generic solution can be obtained in this way.
Dispersive deformations of first integrals (NLS case)

\[ h_f = D_{\text{NLS}} f = f - \frac{\epsilon^2}{12} \left( \left( f_{uuu} + \frac{3}{2u} f_{uu} \right) u_x^2 + 2 f_{uvv} u_x v_x - u f_{uuu} v_x^2 \right) \]

\[ + \epsilon^4 \left\{ \frac{1}{120} \left( \left( f_{uuuu} + \frac{5}{2u} f_{uuu} \right) u_{xx}^2 + 2 f_{uuuv} u_{xx} v_{xx} - u f_{uuuu} v_{xx}^2 \right) \right\} \]

\[ - \frac{1}{80} f_{uuuv} u_{xx} v_x^2 - \frac{1}{48u} f_{uuuvv} v_{xx} u_x^2 - \frac{1}{3456u^3} \left( 30 f_{uuu} - 9 u f_{uuuu} + 12 u^2 f_{5u} + 4 u^3 f_{6u} \right) u_x^4 \]

\[ - \frac{1}{432u^2} \left( -3 f_{uuuv} + 6 u f_{uuuvv} + 2 u^2 f_{5u} v \right) u_x^3 v_x + \frac{1}{288u} \left( 9 f_{uuuu} + 9 u f_{5u} + 2 u^2 f_{6u} \right) u_x^2 v_x^2 \]

\[ + \frac{1}{2160} \left( 9 f_{uuuvv} + 10 u f_{5u} v \right) u_x v_x^3 - \frac{u}{4320} \left( 18 f_{5u} + 5 u f_{6u} \right) v_x^4 \}

+ O(\epsilon^6)

where \[ f_{vv} + u f_{uu} = 0 \]

Remark. Using dispersionless limit of the standard NLS conservation laws one obtains only half of solutions \( f(u,v) \).
Another half from construction of Extended Toda/Extended NLS hierarchy, G.Carlet, B.D., Youjin Zhang 2004
Adding dispersive corrections

Hyperbolic case

\[ r_+ = r_0^+ + c x_+ + \alpha_+ \epsilon^{4/7} U'' (a \epsilon^{-6/7} x_-; b \epsilon^{-4/7} x_+) + \mathcal{O} (\epsilon^{6/7}) \]

\[ r_- = r_0^- + \alpha_- \epsilon^{2/7} U (a \epsilon^{-6/7} x_-; b \epsilon^{-4/7} x_+) + \mathcal{O} (\epsilon^{4/7}) \]

where \( U(X,T) \) is the solution to \( P_I^2 \)
Elliptic case

\[ r_+ = r_+^0 + \gamma \epsilon^{2/5} w \left( \beta \epsilon^{-4/5} (x_+ - x_+^0) \right) + \mathcal{O} \left( \epsilon^{4/5} \right) \]

where

\[ w = w(z) \]

is the tritronquée solution to the Painlevé-I eq.

\[ w'' = 6w^2 - z \]
Thm. (O. Costin et al. 2012) analyticity of the tritronquée solution in the sector $|\arg z| < \frac{4\pi}{5}$
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Thm. (O. Costin et al. 2012) analyticity of the tritronquée solution in the sector \(| \arg z | < \frac{4\pi}{5} \)
NLS, dispersionless NLS, tritronquée solution to $P_l$

$$u = |\psi|^2$$
\[ \psi = \epsilon \cdot \text{Im}(\log \psi)_x \]
About tritronquée solution to Painlevé-I equation

\[ w'' = 6w^2 - z \]

Any solution is meromorphic on the entire complex plane.

Distribution of poles: Theorem of Boutroux (1913)

- Poles of a generic solution to P-I accumulate along the rays

\[ \arg z = \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2 \]
Boutroux ansatz (1913): solutions to P-I

\[ w(z) \simeq \sqrt{z} \wp \left( \frac{4}{5} z^{5/4} \right) \quad \text{for large } |z| \]
Boutroux ansatz (1913): solutions to P-1

\[ w(z) \simeq \sqrt{z} \varphi \left( \frac{4}{5} z^{5/4} \right) \quad \text{for large} \ |z| \]

Hint: do a substitution
Boutroux ansatz (1913): solutions to P-1

\[ w(z) \simeq \sqrt{z} \varphi \left( \frac{4}{5} z^{5/4} \right) \quad \text{for large } |z| \]

Hint: do a substitution

\[ w = z^{1/2} W \]

\[ Z = \frac{4}{5} z^{5/4} \]
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to arrive at
Boutroux ansatz (1913): solutions to P-1

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Hint: do a substitution

\[ w = z^{1/2} W \]
\[ Z = \frac{4}{5} z^{5/4} \]

to arrive at

\[ W'' = 6 W^2 - 1 - \frac{1}{Z} W' + \frac{4}{25} \frac{W}{Z^2} \]
Boutroux ansatz (1913): solutions to P-1

\[ w(z) \simeq \sqrt{z} \varphi \left( \frac{4}{5} z^{5/4} \right) \quad \text{for large } |z| \]

Hint: do a substitution

\[ w = z^{1/2} W \]

\[ Z = \frac{4}{5} z^{5/4} \]

to arrive at

\[ W'' = 6 W^2 - 1 - \frac{1}{Z} W' + \frac{4}{25} \frac{W}{Z^2} \simeq 6 W^2 - 1 \]
2) For any three consecutive rays there exists a unique \textit{tritronquée} solution \( w(z) \) such that the lines of poles truncate along these three rays for large \( |z| \).

Denote \( w_0(z) \) the \textit{tritronquée} solution with lines of poles truncated along the green rays of poles truncated along the green rays.

\[
 w_0(z) \sim -\sqrt{\frac{z}{6}}, \quad |z| \to \infty, \quad |\arg z| < \frac{4\pi}{5} 
\]
Towards rigorous proof: the focusing NLS case:

• inverse spectral transform for the Zakharov - Shabat operator

\[ i\epsilon \frac{d}{dx} + \left( \begin{array}{cc} \lambda & \psi \\ \bar{\psi} & -\lambda \end{array} \right) \]

• semiclassical asymptotics for scattering data

• Deift - Zhou nonlinear steepest descent analysis of the Riemann - Hilbert problem
Towards rigorous proof: the focusing NLS case:

• inverse spectral transform for the Zakharov - Shabat operator

$$i \epsilon \frac{d}{dx} + \begin{pmatrix} \lambda & \psi \\ \bar{\psi} & -\lambda \end{pmatrix}$$

• semiclassical asymptotics for scattering data

• Deift - Zhou nonlinear steepest descent analysis of the Riemann - Hilbert problem  
  Bertola & Tovbis, 2010
Towards rigorous proof: the focusing NLS case:

- inverse spectral transform for the Zakharov - Shabat operator
  \[ i \epsilon \frac{d}{dx} + \begin{pmatrix} \lambda & \psi \\ \bar{\psi} & -\lambda \end{pmatrix} \]

- semiclassical asymptotics for scattering data

- Deift - Zhou nonlinear steepest descent analysis of the Riemann - Hilbert problem
  Bertola & Tovbis, 2010
Remark. Universality and random matrices.

\[ Z_N(t; \epsilon) = \frac{1}{\text{Vol}_N} \int_{N \times N} e^{-\frac{1}{\epsilon} \text{Tr} V(M)} dM \]

Integral over the space of \( N \times N \) Hermitean matrices \( M^* = M \)

\[ V(M) = \frac{1}{2} M^2 - \sum_k t_k \frac{M^{k+1}}{(k+1)!} \]

\[ \text{Vol}_N = \text{Vol} (U(N)/U(1) \times \cdots \times U(1)) \]

\[ N = \frac{x}{\epsilon}, \quad x = \text{‘t Hooft parameter} \quad \Rightarrow \quad Z_N(t; \epsilon) \to Z(x, t; \epsilon) \]
Claim. \( \tau = Z(x, t; \epsilon) \)

is tau function of a solution to Toda hierarchy

i.e., the functions

\[
\begin{align*}
u &= \log \frac{\tau(x+\epsilon)\tau(x-\epsilon)}{\tau^2(x)} \\
v &= \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(x+\epsilon)}{\tau(x)}
\end{align*}
\]

satisfy

\[
\epsilon \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[ (L^{k+1})_+, L \right]
\]

where

\[
L = \Lambda + v(x) + e^{u(x)} \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}
\]

difference Lax operator.
Proof uses representation $Z_N = h_0 h_1 \ldots h_{N-1}$

in terms of monic orthogonal polynomials

$$p_n(x) = x^n + \ldots, \quad \int_{-\infty}^{\infty} p_n(x)p_m(x)e^{-\frac{1}{\epsilon}V(x)}dx = h_n\delta_{mn}$$

For $t = 0$ $p_n(x) \sim H_n(x)$ (Hermite polynomials)

$\Rightarrow$ initial conditions

$$u(x, t = 0, \epsilon) = \log x, \quad v(x, t = 0, \epsilon) = 0$$

Gradient catastrophes=various critical regimes
References


T.Claeys, M.Vanlessen, The existence of a real pole-free solution of the fourth order analogue of the Painlevé-I equation, Nonlinearity 20 (2007) 1163-1184


D.Masoero, A.Raimondo, Semiclassical limit for generalized KdV equation before the time of gradient catastrophe, arXiv:1107.0461


Thank you!
About viscous case

\[ u_t + a(u)u_x = \epsilon [b_1(u)u_{xx} + b_2(u)u_x^2] + \epsilon^2 [c_1(u)u_{xxx} + c_2(u)u_{xx}u_x + c_3(u)u_x^3] + \ldots \]

Critical behaviour (shock wave)

\[ u(x, t; \epsilon) = u_0 + \gamma \epsilon^{1/4} \Gamma \left( \frac{x - x_0 - a_0(t - t_0)}{\alpha \epsilon^{3/4}}, \frac{t - t_0}{\beta \epsilon^{1/2}} \right) + O(\epsilon^{1/2}) \]

Here  \( \Gamma(\xi, \tau) \)  is the logarithmic derivative of the Pearcey function

\[ \Gamma(\xi, \tau) = -2 \frac{\partial}{\partial \xi} \log \int_{-\infty}^{\infty} e^{-\frac{1}{8}(z^4 - 2z^2\tau + 4z\xi)} \, dz \]

B.D., M.Elaeva, 2012

Particular case  \( u_t + a(u)u_x = \epsilon u_{xx} \)  (A.II’In 1985)